Market Microstructure Tutorial
R/Finance 2009

Dale W.R. Rosenthal¹

Department of Finance and
International Center for Futures and Derivatives
University of Illinois at Chicago

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¹daler@uic.edu
Assistant Professor of Finance at UIC
   – Teach Investments; Market Microstructure; Commodities.


Trader/Researcher, Equity Trading Lab, Morgan Stanley, 2000-3.
Market microstructure studies the details of how markets work.

Market microstructure is not “neoclassical” finance.²
- Often directly opposes Efficient Market Hypothesis.

If you believe markets are efficient:
- The details of how markets work are irrelevant since...
- You always get efficient price (less universally-known fee).

Thus microstructure embraces:
- The possibility of short-term alpha; and
- Behavioral effects.

²“So you short one stock...” omits a lot.
What I Cover in a Microstructure Course

When I teach microstructure, I cover many topics:

1. Market Types
2. Orders and Quotes
3. Trades and Traders
4. Market Structures
5. Roll and Sequential Trade Models ← Play with some
6. Strategic Trader and Inventory Models ← of these today.
7. Prices, Sizes, and Times
8. Liquidity and Transactions Costs
9. Market Metrics Across Time
10. Electronic Markets
11. Electronic Trading Tools and Strategies
Today we’ll consider models for microstructure phenomena.

For these models, need to adopt a different perspective.

Models are simple; lets us conduct controlled experiments.
  - Eliminate all but one or two major factors.
  - Question: Does this model reproduce real-life features?
  - If so: factors we are considering probably matter.
  - We may even have an idea about how those factors matter.

Less confusion about what matters helps build better models.

Newest research combining such models with time series.

*All models are wrong; some models are useful.*

— George Box
We will examine two asymmetric information models.
- **Asymmetry**: Trades may contain private information.
- **Sequential trade**: independent sequence of traders.
  - Traders: informed (know asset value) or not; trade once.
  - Single market maker; learns information by trading.
- **Strategic trader**: one informed trader can trade many times.
  - Informed trader considers own impact on later trades.
  - Uninformed (noise) traders also submit orders.
  - Single MM sees combined order, sets price, fills order.

Some slides are for you to study later; I’ll skip those.
Glosten-Milgrom (1985) Model
Glosten and Milgrom (1985) Model

- Most famous sequential trade model.
- MM quotes a bid $B$ and ask $A$.
- Security has value $V = \underline{V}$ or $\overline{V}$, $\underline{V} < \overline{V}$.
- At time $t = 0$, informed traders (only) learn $V$.
- Time is discretized.
- Trades are for one unit, occur at each time step.
- MM has infinite capital: no inventory/bankruptcy concerns.
Glosten-Milgrom Model: Setup

\[ V \sim \begin{cases} \overline{V} & \text{w.p. } 1 - \delta \\
V & \text{w.p. } \delta \end{cases} \]

- Trader type \( T \sim \begin{cases} \text{Informed} & \text{w.p. } \mu \\
\text{Uninformed} & \text{w.p. } 1 - \mu \end{cases} \).

- Traders take action \( S \) (buy/sell from MM), one at a time.

- Informed traders: \( S \sim \begin{cases} \text{buy} & \text{if } V = \overline{V} \\
\text{sell} & \text{if } V = V \end{cases} \).

- Uninformed traders: \( S \sim \begin{cases} \text{buy} & \text{w.p. } 1/2 \\
\text{sell} & \text{w.p. } 1/2 \end{cases} \).
Glosten-Milgrom Model: Event Trees

- $V$ = security value ($\overline{V}, V$)
- $T$ = trader type (Informed/Uninformed)
- $S$ = trader’s side (buy/sell)

$t = 0$

$t = 1, 2, \ldots$
What is \( P(\text{buy}) \), \( P(\text{sell}) \) after one trade?

\[
P(\text{buy}) = P(\text{buy}|\mathcal{V})P(\mathcal{V}) + P(\text{buy}|\bar{\mathcal{V}})P(\bar{\mathcal{V}})
\]

\[
= \frac{(1 - \mu)\delta + (1 + \mu)(1 - \delta)}{2} = \frac{1 + \mu(1 - 2\delta)}{2}
\]

\[
P(\text{sell}) = P(\text{sell}|\mathcal{V})P(\mathcal{V}) + P(\text{sell}|\bar{\mathcal{V}})P(\bar{\mathcal{V}})
\]

\[
= \frac{(1 + \mu)\delta + (1 - \mu)(1 - \delta)}{2} = \frac{1 - \mu(1 - 2\delta)}{2}
\]
Glosten-Milgrom Model: Likelihood of $\overline{V}$, $\overline{V}$

After one trade, we have information via Bayes’ Theorem:

\[
P(\overline{V}|\text{buy}) = \frac{P(\text{buy}|\overline{V})P(\overline{V})}{P(\text{buy})} = \frac{(1 - \delta)(1 + \mu)}{1 + \mu(1 - 2\delta)} \quad (5)
\]

\[
P(\overline{V}|\text{buy}) = \frac{P(\text{buy}|\overline{V})P(\overline{V})}{P(\text{buy})} = \frac{\delta(1 - \mu)}{1 + \mu(1 - 2\delta)} \quad (6)
\]

\[
P(\overline{V}|\text{sell}) = \frac{P(\text{sell}|\overline{V})P(\overline{V})}{P(\text{sell})} = \frac{(1 - \delta)(1 - \mu)}{1 - \mu(1 - 2\delta)} \quad (7)
\]

\[
P(\overline{V}|\text{sell}) = \frac{P(\text{sell}|\overline{V})P(\overline{V})}{P(\text{sell})} = \frac{\delta(1 + \mu)}{1 - \mu(1 - 2\delta)} \quad (8)
\]
Glosten-Milgrom Model: Bid, Ask, Spread

- If competition narrows profit to 0, before trading...
- \( A = E(V|\text{buy}) \) and \( B = E(V|\text{sell}) \).

\[
A = VP(V|\text{buy}) + VP(V|\text{buy}) \\
= \frac{V}{1 + \mu(1 - 2\delta)} \delta(1 - \mu) + \frac{V}{1 + \mu(1 - 2\delta)} (1 - \delta)(1 + \mu) \\
(9)
\]

\[
B = VP(V|\text{sell}) + VP(V|\text{sell}) \\
= \frac{V}{1 - \mu(1 - 2\delta)} \delta(1 + \mu) + \frac{V}{1 - \mu(1 - 2\delta)} (1 - \delta)(1 - \mu) \\
(11)
\]

\[
A - B = \frac{4(1 - \delta)\delta\mu(V - V)}{1 - \mu^2(1 - 2\delta)^2} \\
(13)
\]
Glosten-Milgrom Model: Updating Bids and Asks

- After each trade, Bayesian update of beliefs about $V$.
- Idea: How often would we see these trades if $V = \bar{V}$ vs. $\underline{V}$?
- With these ideas, we can update the bid and ask prices.
- Expected bid and ask after $k$ buys and $\ell$ sells:

\[
A_{k+\ell} = \frac{V \delta (1 - \mu)^{k+1} (1 + \mu)\ell + \bar{V} (1 - \delta)(1 + \mu)^{k+1}(1 - \mu)\ell}{(1 - \delta)(1 + \mu)^{k+1}(1 - \mu)\ell + \delta(1 - \mu)^{k+1}(1 + \mu)\ell} \tag{14}
\]

\[
B_{k+\ell} = \frac{V \delta (1 - \mu)^{k} (1 + \mu)^{\ell+1} + \bar{V} (1 - \delta)(1 - \mu)^{k}(1 + \mu)^{\ell+1}}{(1 - \delta)(1 + \mu)^{k}(1 - \mu)^{\ell+1} + \delta(1 - \mu)^{k}(1 + \mu)^{\ell+1}} \tag{15}
\]
Glosten-Milgrom Model: Simulation Example

- One simulation for $\mu = 0.3$, $\delta = 0.5$; example bids and asks.
- Simple case: $V = \overline{V} = 2$ (versus $\underline{V} = 1$).
- Can see price impact — especially for sequence of orders.
Glosten-Milgrom Model: Simulation Averages

- Simulate to find the average bid and ask.
- 10,000 simulations; tried $\mu = 0.3, 0.5$, $\delta = 0.3, 0.5$.
- Simple case: $V = \overline{V} = 2$ (versus $V = 1$).

Simple case: $V = \overline{V} = 2$ (versus $V = 1$).
Glosten-Milgrom Model: Other Results

- Basic idea: Spreads exist due to adverse selection.
- Buys/sells are unbalanced; but, price series is a martingale.
- Orders are serially correlated: buys tend to follow buys.
  - This and the preceding line seem contradictory.
  - Difference is akin to Pearson $\rho$ versus Kendall $\tau$.
- Trades have price impact: a buy increases $B$ and $A$.
- Spreads tend to decline over time as MMs figure out $V$.
- Bid-ask may be such that market effectively shuts down\(^3\)
- If uninformed were price sensitive, spreads would be wider.
- Code in THE SECRET DIRECTORY (glosten-milgrom.r).
- Fun: Add very rare third trader, govt, who always buys at $V$.
  - Stunning: still converging after 50,000 trades.

\(^3\)We find ourselves in the game-theoretic Paradox of Trade.
Kyle (1985) Model
Kyle (1985) proposed a model with a single informed trader. The informed trader:
- Considers price impact in setting trade size;
- Learns security’s terminal value \( v \); and,
- Submits order for quantity \( x \).

Liquidity ("noise") traders submit net order \( u \).

The single market maker (MM):
- Observes total order \( y = x + u \);
- Makes up the difference; and,
- Sets the market clearing price \( p \).

All trades happen at one price; no bid-ask spread.

All trading occurs in one period.
The informed trader would like to trade aggressively.

Q1: What sort of function maps $v$ to order size?
A: Linear function? (Yes.)

MM knows larger net orders are more likely to be informed.
Thus MM sets price increasing in net order size.

Q2: What sort of function maps order size to MM’s price?
A: Linear function? (Yes.)
One-Period Kyle Model: Setup

Security value  \( v \sim N(p_0, \Sigma_0) \)  \( (18) \)

Noise order  \( u \sim N(0, \sigma_u^2) \)  \( u \perp v \)  \( (19) \)

MM assumes: informed order  \( x = \beta v + \alpha \)  \( (20) \)

Net order  \( y = x + u \)  \( (21) \)

Informed assumes: trade price  \( p = \lambda \{ y + \mu \} \)  \( \text{illiquidity} \)  \( (22) \)

Informed trader profit  \( \pi = (v - p)x \)  \( (23) \)

\[ = (v - \lambda(x + u) - \mu)x \]  \( (24) \)
If we combine the previous formulæ, we get a little further:

\[
E(\pi) = E((v - \lambda(x + u) - \mu)x) = (v - \lambda x - \mu)x
\]

\[E(u) = 0; x \text{ non-random}\]

\[x^* = \arg\max_{x \in \mathbb{R}} E(\pi) = \frac{v - \mu}{2\lambda} \quad \text{if } \lambda > 0\] (25)

\[
\Rightarrow \beta = \frac{1}{2\lambda}, \alpha = \frac{-\mu}{2\lambda}
\] (26)

\[
E(y) = \alpha + \beta E(v) = \frac{-\mu}{2\lambda} + \frac{p_0}{2\lambda}
\] (27)

\[
\text{Var}(y) = \text{Var}(x) + \text{Var}(u) = \beta^2 \text{Var}(v) + \text{Var}(u)
\]

\[= \beta^2 \Sigma_0 + \sigma_u^2\]

\[
\text{Cov}(y, v) = \text{Cov}(\alpha + \beta v, v) = \beta \Sigma_0
\] (28) (29) (30) (31)
Now solve for the linear MM pricing and trader order parameters.

- MM earns no expected profit\(^4\), prices trade at \( p = E(v|y) \).
- Since \( v, y \) normal, form of \( E(v|y) \) is like linear regression.

\[
p = E(v|y) = E(v) + \frac{\text{Cov}(v, y)}{\text{Var}(y)} (y - E(y)) \tag{32}
\]

\[
= p_0 + \frac{\beta \Sigma_0}{\sigma_u^2 + \beta^2 \Sigma_0} (y - \alpha - \beta p_0) \tag{33}
\]

- Use (33), (27), and (22) to solve for \( \alpha, \beta, \mu, \lambda \):

\[
\alpha = p_0 \frac{\sigma_u}{\sqrt{\Sigma_0}}; \quad \beta = \frac{\sigma_u}{\sqrt{\Sigma_0}}; \quad \mu = p_0; \quad \lambda = \frac{\sqrt{\Sigma_0}}{2\sigma_u}. \tag{34}
\]

\(^4\)Due to competition, again.
One-Period Kyle Model: MM Price; Informed Order, Profit

MM trade price 
\[ p = E(v|y) = \lambda y + \mu = \frac{\sqrt{\Sigma_0}}{2\sigma_u} \cdot y + p_0 \] (35)

Value uncertainty
\[ \text{Var}(v|y) = \frac{\sigma_u^2 \Sigma_0}{\sigma_u^2 + \beta^2 \Sigma_0} = \frac{\Sigma_0}{2} \] (36)

Informed order
\[ x^* = \beta v + \alpha = \frac{(v - p_0)\sigma_u}{\sqrt{\Sigma_0}} \] (37)

Expected profit
\[ E(\pi) = (v - \lambda x - \mu)x = \frac{(v - p_0)^2 \sigma_u}{2\sqrt{\Sigma_0}} \] (38)
What can we learn from the one-period Kyle model?

- Trade price linear in net order size, security volatility.
- Trade price inverse to noise order volatility.
- Informed order linear in security’s deviation from mean.
- Expected profit quadratic in security’s deviation from mean.
  - Large deviations matter much more than small deviations\(^5\).
- Informed order, expected profit linear in noise order volatility\(^6\).
- \(\text{Var}(\nu | y) = \frac{\Sigma_0}{2} \implies \text{Half of information}^7 \text{ leaks after one trade.}\)
- Negative net order (i.e. \(u < -x\)) yields negative price. (!)

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\(^5\)Except data errors and adverse selection will hurt you.
\(^6\)Use uninformed orders to hide.
\(^7\)Information in a Fisher sense.
Finally: Consider the illiquidity parameter.

- Illiquidity parameter $\lambda = \sqrt{\frac{\Sigma_0}{2\sigma_u}}$.
- $\sqrt{\Sigma_0}/\sigma_u$ is ratio of volatilities.
  - Value uncertainty vs. noise order uncertainty.
- $\lambda y = \sqrt{\Sigma_0 \frac{y}{2\sigma_u}}$: like liquidity risk:
  - Scaled by volatility of security; and,
  - $y/\sigma_u$ is similar/proportional to percentage of volume.
- Nice: Demanding liquidity has a cost.
- Full course covers more such ideas.
Kyle also discussed a multi-period model.

Slice time $t \in \{0, 1\}$ into $N$ bins, $n = 1, \ldots, N$:

\begin{align*}
\text{Time:} & \quad \Delta t_n = 1/N, \ t_n = n/N. \\
\text{Noise order:} & \quad \Delta u_n \sim N(0, \sigma_u^2 \Delta t_n). \\
\text{Informed order:} & \quad \Delta x_n = \beta_n (v - p_{n-1})/N. \\
\text{Price change:} & \quad \Delta p_n = \lambda_n (\Delta x_n + \Delta u_n).
\end{align*}

E(Later profit):

\begin{equation}
E(\pi_n|p_t < n) = \alpha_{n-1} (v - p_{n-1})^2 + \delta_{n-1}
\end{equation}

Competition $\Rightarrow p_n = E(v|\Delta x_1 + \Delta u_1, \ldots, \Delta x_n + \Delta u_n)$

Informed orders not autocorrelated, by construction.
Use the following definitions:

Strategy: \( X = (x_1, \ldots, x_N) \). \hfill (44)

Pricing rule: \( P = (p_1, \ldots, p_N) \). \hfill (45)

\( X \) chosen to always maximize expected future profit:

\[
E(\pi_n(X, P)|v, p_1, \ldots, p_{n-1}) \geq E(\pi_n(X', P')|v, p_1, \ldots, p_{n-1}) \quad n = 1, \ldots, N. \hfill (46)
\]
Model dynamics are given by difference equations for $n = 1, \ldots, N$:

$$\delta_{n-1} = \delta_n + \alpha_n \lambda_n^2 \sigma_u^2 \Delta t_n; \quad (47)$$

$$\alpha_{n-1} = \frac{1}{4\lambda_n (1 - \alpha_n \lambda_n)}; \quad (48)$$

$$\beta_n \Delta t_n = \frac{1 - 2\alpha_n \lambda_n}{2\lambda_n (1 - \alpha_n \lambda_n)}; \quad (49)$$

$$\lambda_n = \frac{\beta_n \Sigma_n}{\sigma_u^2}; \quad \text{and,}$$

$$\Sigma_n = (1 - \beta_n \lambda_n \Delta t_n) \Sigma_{n-1}. \quad (50)$$

$\lambda_n$ is the middle root of the cubic equation:

$$(1 - \lambda_n^2 \sigma_u^2 \Delta t_n / \Sigma_n)(1 - \alpha_n \lambda_n) = \frac{1}{2}. \quad (52)$$
Solving for the model dynamics is a bit crusty:

1. Guess $\Sigma_N$ (call the guess $\Sigma^*_N$).

2. Use $\alpha_N = \delta_N = 0$ to get $\lambda_N = \frac{\sqrt{\Sigma^*_N}}{\sigma_u \sqrt{2\Delta t_N}}$.


4. Solve for $\beta_n$ and $\Sigma^*_{n-1}$.

5. Find $\alpha_{n-1}$ for $\lambda_n$.

6. Solve (52)$^8$, using middle root for $\lambda_{n-1}$.

7. $n = n - 1$; if $n > 0$, go to step 4.

8. If $|\Sigma^*_0 - \Sigma_0| > \epsilon$: try another $\Sigma^*_N$, go to step 1.

9. Solve for $\beta_0$

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$^8$Numerically or via Cardano’s formula.
Multi-Period Kyle Model: Simulation

- We can simulate the Kyle model to see its behavior.
- Use $p_0 = 2$, $\Sigma_0 = 0.4$, and $\sigma_u = 0.5$.
- Run one simulation. What do we get?
- The parameter evolution is not so surprising.
- The action evolution is more illuminating.
Multi-Period Kyle Model: Evolution of Parameters I

- Delta parameter over time
- Alpha parameter over time
- Beta parameter over time

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Multi-Period Kyle Model: Evolution of Actions

- Orders
- Time
- Trade Prices
- True Price

A few details nobody has previously noted:\(^{10}\):

- Informed orders\(^{11}\) are larger after negative uninformed trades.
- Informed orders decrease after larger net orders, price moves.
  - Recall: one leads to the other; confounding lives here.
- Informed order size increases slightly with time.
- Trade price moves toward the true value.
- Trade price may not converge to true value by end of trading.

Code in THE SECRET DIRECTORY (kyle.r).

\(^{10}\)As with the Glosten-Milgrom model, I have yet to see plots from anybody else who has simulated the Kyle model.

\(^{11}\)Informed trades are shown as 'i's; uninformed trades as 'u's.
If You Want More: to Read

- Journals (and associated societies):
  - Journal of Financial Markets
  - Journal of Business and Economic Statistics/ASA
  - Journal of Financial Econometrics/SoFiE
  - Journal of Financial Economics
  - Journal of Financial and Quantitative Analysis
  - Review of Financial Studies/SFS

- Books:
  - O’Hara, *Market Microstructure Theory*
  - Harris, *Trading and Exchanges*
  - Hasbrouck, *Empirical Market Microstructure*
  - Weisberg, *Applied Regression Analysis*
  - Montgomery, *Design and Analysis of Experiments*
  - McCullagh and Nelder, *Generalized Linear Models*
  - Box, Jenkins, Reinsel, *Time Series Analysis*
  - Osborne and Rubinstein, *A Course in Game Theory*
If You Want More: to Interact With

- **Seminars (times may change next year)**

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- **Center Events**: UIC ICFD, NWU Zell, UofC Stevanovich

- **Conferences**: ASSA, SoFiE, Oxford-Man Institute
If You Want: to Support Work Like This

- Talk to academics at the conference; some glad to consult.
- Take courses through UIC External Ed:
  - Market Microstructure and Electronic Trading
  - Commodities, Energy, and Related Markets
  - Fixed Income/Structured Products
  - Empirical Methods for Finance
  - Univariate and Mutivariate Time Series Analysis
- Donate to the ICFD.