# Portfolio optimization with CVaR budgets

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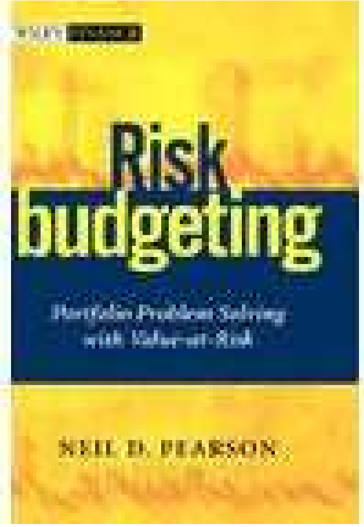
# History



Risk budgets:

- Standard tool to quantify risk allocation;
- Previous research: non-normality return series, CVaR: PerformanceAnalytics.
- Instrument to adjust marginally portfolios;
- This research: Use of risk budgets as objective and/or constraint in portfolio allocation styles: PortfolioAnalytics;
- Collaborative: Peter Carl & Brian Peterson, David Ardia, Christophe Croux.



# Motivation equity/bonds portfolio



- Bender, Briand, Nielsen, Stefek (JPM, Winter 2010):

  *"Traditional approaches to structuring policy portfolios for strategic asset allocation have not provided the full potential of diversification. Portfolios based on a 60/40 allocation between equities and bonds remain volatile and dominated by equity risk."*

- Minimum risk portfolios tend to be dominated by bond risk and have lower expected returns.

# Motivation equity/bonds portfolio

- Bender, Briand, Nielsen, Stefek (JPM, Winter 2010):

  *"Traditional approaches to structuring policy portfolios for strategic asset allocation have not provided the full potential of diversification. Portfolios based on a 60/40 allocation between equities and bonds remain volatile and dominated by equity risk."*

- Minimum risk portfolios tend to be dominated by bond risk and have lower expected returns.

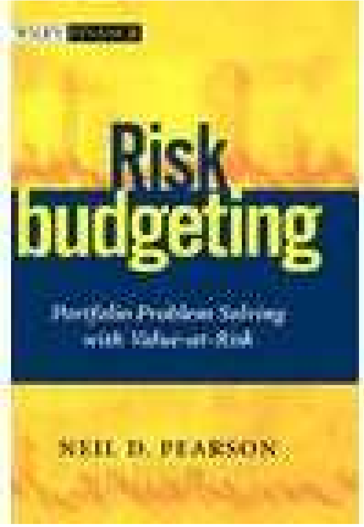
- Optimize the risk allocation directly in the portfolio strategy. Examples:
  - ✓ A 60/40 risk allocation portfolio or an equal-risk portfolio
  - ✓ The most risk diversified portfolio subject to return/risk targets.

# Outline



Risk budgets:

- Review on risk budgets;
- Use of risk budgets as objective and/or constraint in portfolio allocation styles.
- Illustrations:
  - ✓ Static bond-equity portfolio, R code (DEoptim, see also Guy Yollin's slides RFinance 2009);
  - ✓ Dynamic 4 assets.



# A primer on risk budgets

# VaR budget on 60/40 portfolio



```
> library(PortfolioAnalytics)
> data(indexes)
> head(indexes[,1:2])
              US Bonds US Equities
1980-01-31 -0.027189977      0.0610
1980-02-29 -0.066876898      0.0031
1980-03-31  0.005275587     -0.0987
1980-04-30  0.099246261      0.0429
1980-05-31  0.000000000      0.0562
1980-06-30  0.060511451      0.0296
```

# VaR budget on 60/40 portfolio



```
> apply(indexes[,1:2],2,'mean')
   US Bonds US Equities
0.006916187 0.008238420
> apply(indexes[,1:2],2,'sd')
   US Bonds US Equities
 0.01810161  0.04476569
```

# VaR budget on 60/40 portfolio



```
> w6040 <- c(0.4,0.6)
> library(PerformanceAnalytics)
> VaR(R=indexes[,1:2], weights=w6040,
+       portfolio_method="component")
$MVaR
[1,] 0.04336715
$contribution
     US Bonds   US Equities
-0.0002303964  0.0435975440
$pct_contrib_MVaR
    US Bonds  US Equities
-0.005312695  1.005312695
```

- $C_i\text{VaR} = w_i \dfrac{\partial \text{VaR}(w)}{\partial w_i}$
- Gouriéroux, Laurent and Scaillet (2000):

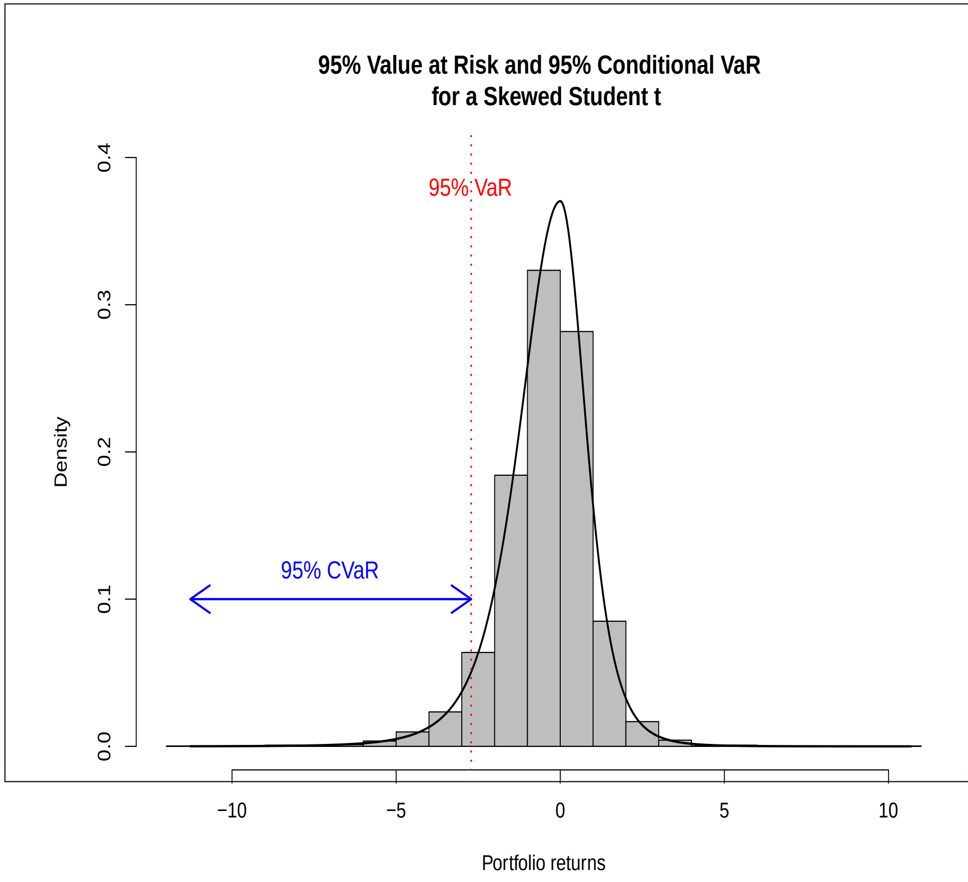
$$C_i\text{VaR} = -E[w_i r_i | r_p = -\text{VaR}]$$

- Estimation:
  - ✓ Simulation
  - ✓ Explicit formulae Cornish-Fisher estimator (Boudt, Peterson and Croux, 2008; Peterson and Boudt, 2008).

Pearson [2002, p.7]: *"Value-at-risk has some well known limitations, and it may be that some other risk measures eventually supplants value-at-risk in the risk budgeting process."*

CVaR:

- Coherent risk measure (most notably: subadditive);
- Less incentive to load on the tail risk below the VaR used.



# CVaR budgets



- $C_i\text{CVaR} = w_i \frac{\partial \text{CVaR}(w)}{\partial w_i}$
- Scaillet (2002):

$$C_i\text{CVaR} = -E[w_i r_i | r_p \leq -\text{VaR}]$$

- Estimation:
  - ✓ Simulation
  - ✓ Explicit formulae Cornish-Fisher estimator (Boudt, Peterson and Croux, 2008).

```
> ES(R=indexes[,1:2], weights=w6040,
+        portfolio_method="component")
$MES
[1,] 0.07725177
$contribution
    US Bonds  US Equities
-0.001066194  0.078317964
$pct_contrib_MES
   US Bonds US Equities
-0.01380155  1.01380155
```

# CVaR in portfolio allocation



- As an objective: Minimum CVaR portfolio (E.g. Rockafellar and Uryasev, 2000)
- As a constraint: Min CVaR/SD portfolio under CVaR constraints (E.g. Alexander and Baptista, 2004, Krokhmal, Palmquist and Uryasev, 2002).
- Why? Better risk measure + convex function of portfolio weights (easier to optimize).

# Min CVaR portfolio



```
> library(DEoptim)
> obj <- function(w) {
+  if (sum(w) == 0) { w <- w + 1e-2 }
+  w <- w / sum(w)
+  ES(R=indexes[,1:2],weights = w)$MES
+ }
> out <- DEoptim(fn = obj, lower = rep(0, 2),
+ upper = rep(1, 2), DEoptim.control(itermax=50))
> wstar <- out$optim$bestmem
> wMinCVaR <- wstar / sum(wstar)
> print(wMinCVaR)
  US Bonds US Equities
0.96443348 0.03556652
```

```
> ES(R=indexes[,1:2], weights=wMinCVaR,
+ portfolio_method="component")
$MES
[1,] 0.01102894
$contribution
    US Bonds  US Equities
0.0106366796 0.0003922610
$pct_contrib_MES
   US Bonds US Equities
 0.96443349  0.03556651
```

# CVaR budgets



| | Weight allocation | | Risk allocation | |
|---|---|---|---|---|
| style | bond | equity | bond | equity |
| 60/40 weight | 0.40 | 0.6 | -0.01 | 1.01 |
| 60/40 risk alloc | 0.84 | 0.16 | 0.40 | 0.60 |
| Min CVaR Conc | 0.86 | 0.14 | 0.50 | 0.50 |
| Min CVaR | 0.96 | 0.04 | 0.96 | 0.04 |

# CVaR budgets as objective or constraint in portfolio allocation

- Risk contribution $C_i\text{CVaR}(w) = w_i \frac{\partial \text{CVaR}(w)}{\partial w_i}$

- Litterman (1999): Hot Spots$^{TM}$ and hedges. Risk budgets as ex post instrument to adjust marginally portfolios.

- Keel and Ardia (2009):

  1. Only precise for infinitesimal changes, poor approximations for realistic reallocations.
  2. Assume changing a single position keeping fixed all other positions $><$ full investment constraint.

# Proposal I: Risk budget constraints



- Avoid downside risk concentration by:

$$l_i \leq \%C_i\text{CVaR} \equiv \frac{C_i\text{CVaR}}{\text{CVaR}} \leq u_i$$

- Min CVaR portfolio with
  - ✓ 60/40 risk allocation constraint
  - ✓ equal risk allocation constraint.

# 60/40 risk allocation portfolio



```
> obj <- function(w) {
+   if (sum(w) == 0) { w <- w + 1e-2 }
+   w <- w / sum(w)
+   CVaR <- ES(R=indexes[,1:2],weights = w)
+   tmp1 <- CVaR$MES
+   tmp2 <- max(CVaR$pct_contrib_ES
+                - c(0.405, 0.605) , 0)
+   tmp3 <- max(c(0.395, 0.595) -
+                CVaR$pct_contrib_ES , 0)
+   out  <- tmp1 + 1e3 * tmp2 + 1e3 * tmp3
+ }
```

# 60/40 risk allocation portfolio



```
> out <- DEoptim(fn = obj, lower = rep(0, 2),
+ upper = rep(1, 2), DEoptim.control(itermax=50))
> wstar <- out$optim$bestmem
> w6040riskalloc <- wstar / sum(wstar)
> print(w6040riskalloc)
  US Bonds US Equities
0.8382035 0.1617965
```

# 60/40 risk allocation portfolio



```
> ES(R=indexes[,1:2], weights=w6040riskalloc,
+ portfolio_method="component")
$MES
[1,] 0.01400341
$contribution
   US Bonds US Equities
0.005671224 0.008332185
$pct_contrib_MES
   US Bonds US Equities
  0.4049888   0.5950112
```

# Special case: Equal-risk portfolio



- Min CVaR with equal-risk constraint

$$\%C_i\text{CVaR}(w) = 1/N \qquad (i = 1, \ldots, N)$$

- In this portfolio:

$$\frac{w_i}{w_j} = \frac{\partial \text{CVaR}/\partial w_j}{\partial \text{CVaR}/\partial w_i}.$$

Downweights “hot spots”: positions for which a marginal

decrease in weight leads to a large reduction in CVaR.

# Proposal II: Risk budget objective



- Avoid downside risk concentration by:

$$\min_w \max_i C_i \text{CVaR}(w)$$

# Proposal II: Risk budget objective



- Avoid downside risk concentration by:

$$\min_w \max_i C_i \text{CVaR}(w)$$

- Objective trades off Risk Minimization and Risk Diversification, since:

$$\max_i C_i \text{CVaR} = \text{CVaR} \max\{\%C_1 \text{CVaR}, \ldots, \%C_N \text{CVaR}\}$$

# Relation with equal-risk portfolio



- If the set of equal-risk portfolios is non-empty and the minimum CVaR concentration portfolio has a unique optimum. Then the minimum CVaR concentration portfolio criterion is equivalent to:

$$\min_w \text{CVaR}(w)$$

$$s.t.\ \%C_1\text{CVaR} = \ldots = \%C_N\text{CVaR}$$

- But computationally more simple and has also a solution if there is no equal-risk portfolio.

# Min CVaR Concentration portfolio



```
> obj <- function(w) {
+  if (sum(w) == 0) { w <- w + 1e-2 }
+  w <- w / sum(w)
+  CVaR <- ES(R=indexes[,1:2],weights = w)
+  out <- max(CVaR$contribution)
+ }
> out <- DEoptim(fn = obj, lower = rep(0, 2),
+  upper = rep(1, 2),DEoptim.control(itermax=50))
> wstar <- out$optim$bestmem
> wMinCVaRConc <- wstar / sum(wstar)
> print(wMinCVaRConc)
 US Bonds US Equities
0.8584465 0.1415535
```

# Min CVaR Concentration portfolio



```
> ES(R=indexes[,1:2], weights=wMinCVaRConc,
+ portfolio_method="component")
$MES
[1,] 0.01315665
$contribution
   US Bonds US Equities
0.006578325 0.006578323
$pct_contrib_MES
   US Bonds US Equities
  0.5000001   0.4999999
```

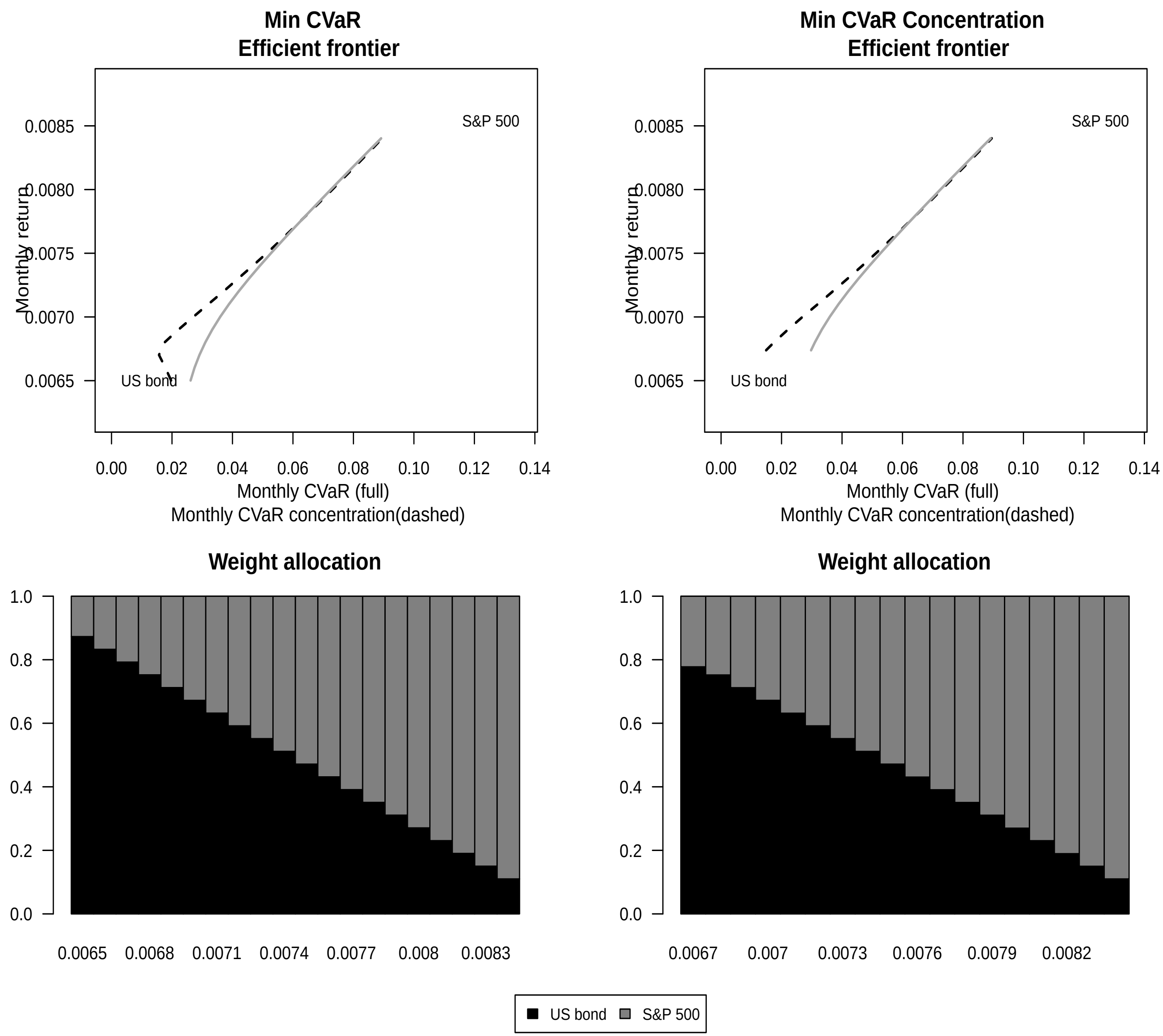
# Overview portfolio allocations



| | Weight allocation | | Risk allocation | |
|---|---|---|---|---|
| style | bond | equity | bond | equity |
| 60/40 weight | 0.40 | 0.6 | -0.01 | 1.01 |
| 60/40 risk alloc | 0.84 | 0.16 | 0.40 | 0.60 |
| Min CVaR Conc | 0.86 | 0.14 | 0.50 | 0.50 |
| Min CVaR | 0.96 | 0.04 | 0.96 | 0.04 |

# Adding a return target: Efficient frontier





# Dynamic portfolio allocation

# Dynamic investment strategies

- We consider the following investment strategies, quarterly rebalancing, 4 assets:

  1. Benchmark strategies:
     - ✓ "Equal Weight"
     - ✓ "Min CVaR"
     - ✓ "Min CVaR + 40% Position Limit"
     - ✓ "Min CVaR + EW return target"
  2. Strategies that use CVaR budgets:
     - ✓ "Min CVaR + 40% Perc CVaR Alloc Limit"
     - ✓ "Min CVaR Conc"
     - ✓ "Min CVaR Conc + EW return target".

## Summary statistics data:

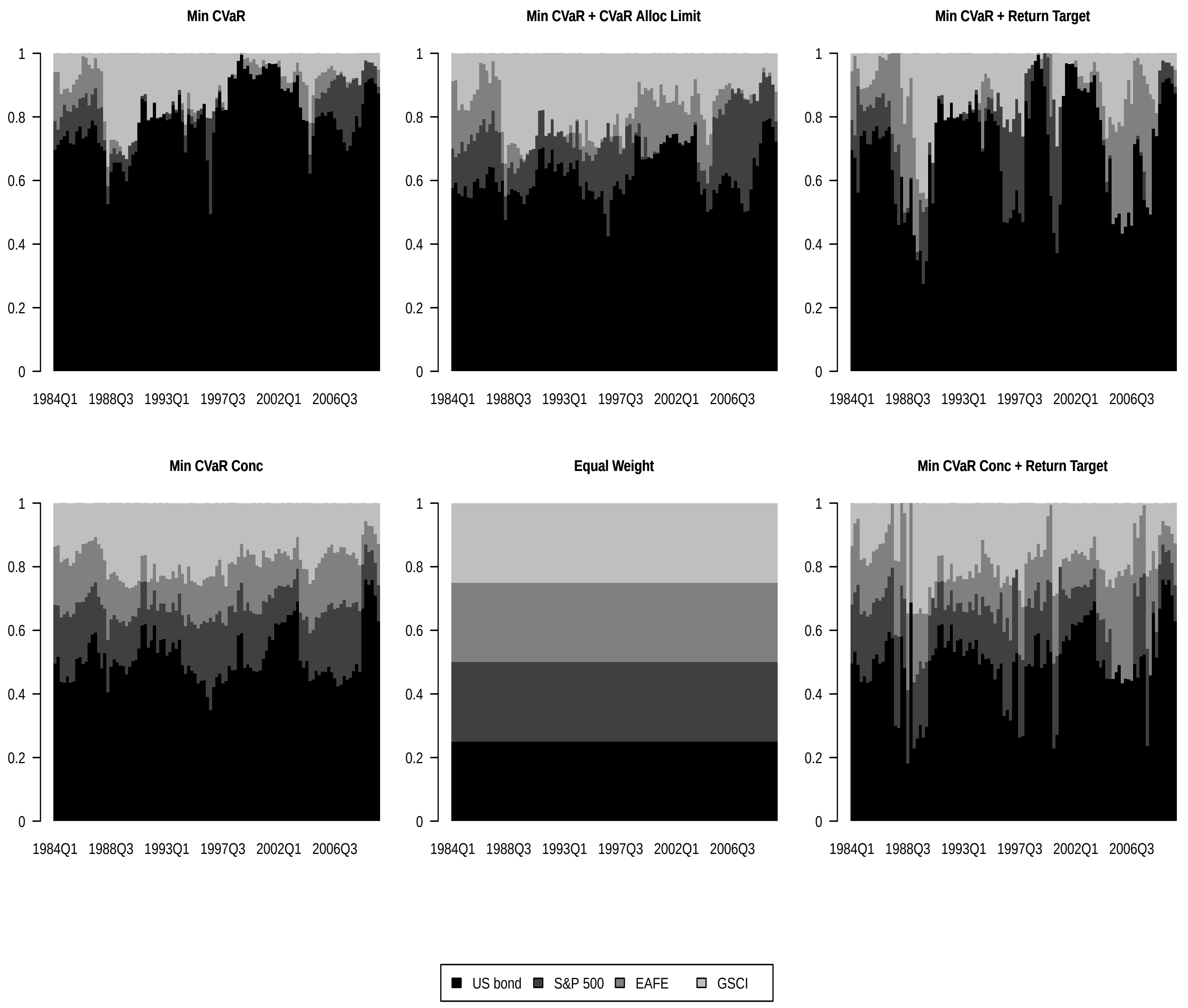
4 assets: Merrill Lynch US bond, S&P 500, MSCI EAFE and S&P GSCI

Real monthly returns Jan 1976-December 2009, total return indices

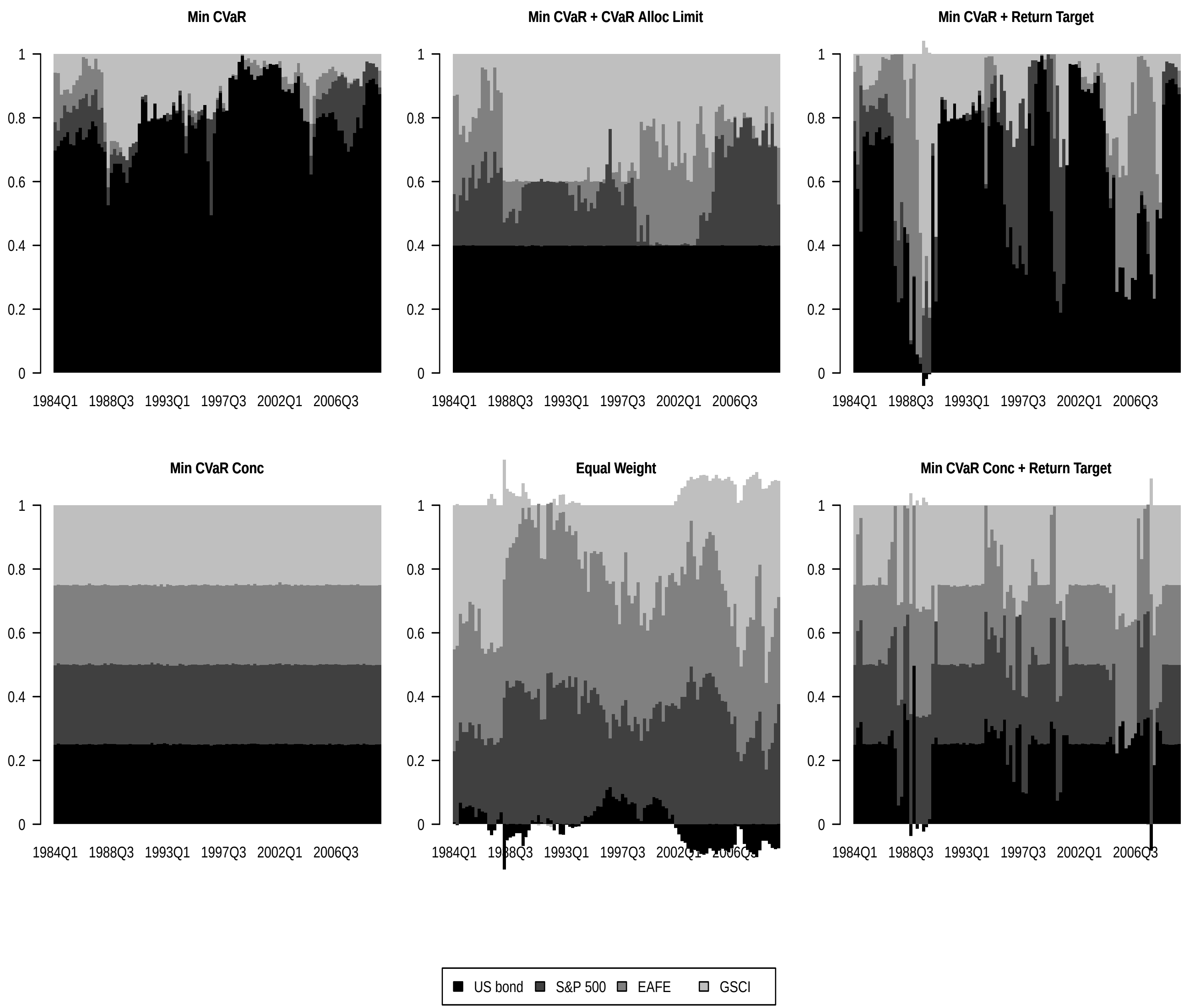
| | US bond | S&P 500 | MSCI EAFE | S&P GSCI |
|---|---|---|---|---|
| Mean (in %) | 0.32 | 0.52 | 0.39 | 0.10 |
| StdDev (in %) | 1.86 | 4.46 | 4.98 | 5.50 |
| Historical 95% CVaR (in %) | 3.64 | 10.64 | 12.46 | 13.58 |

Quarterly rebalanced based on time-varying conditional moment estimates (EWMA mean, GARCH volatility, rolling 8 year correlation, coskewness and cokurtosis).

# Weight allocation:



# CVaR allocation:

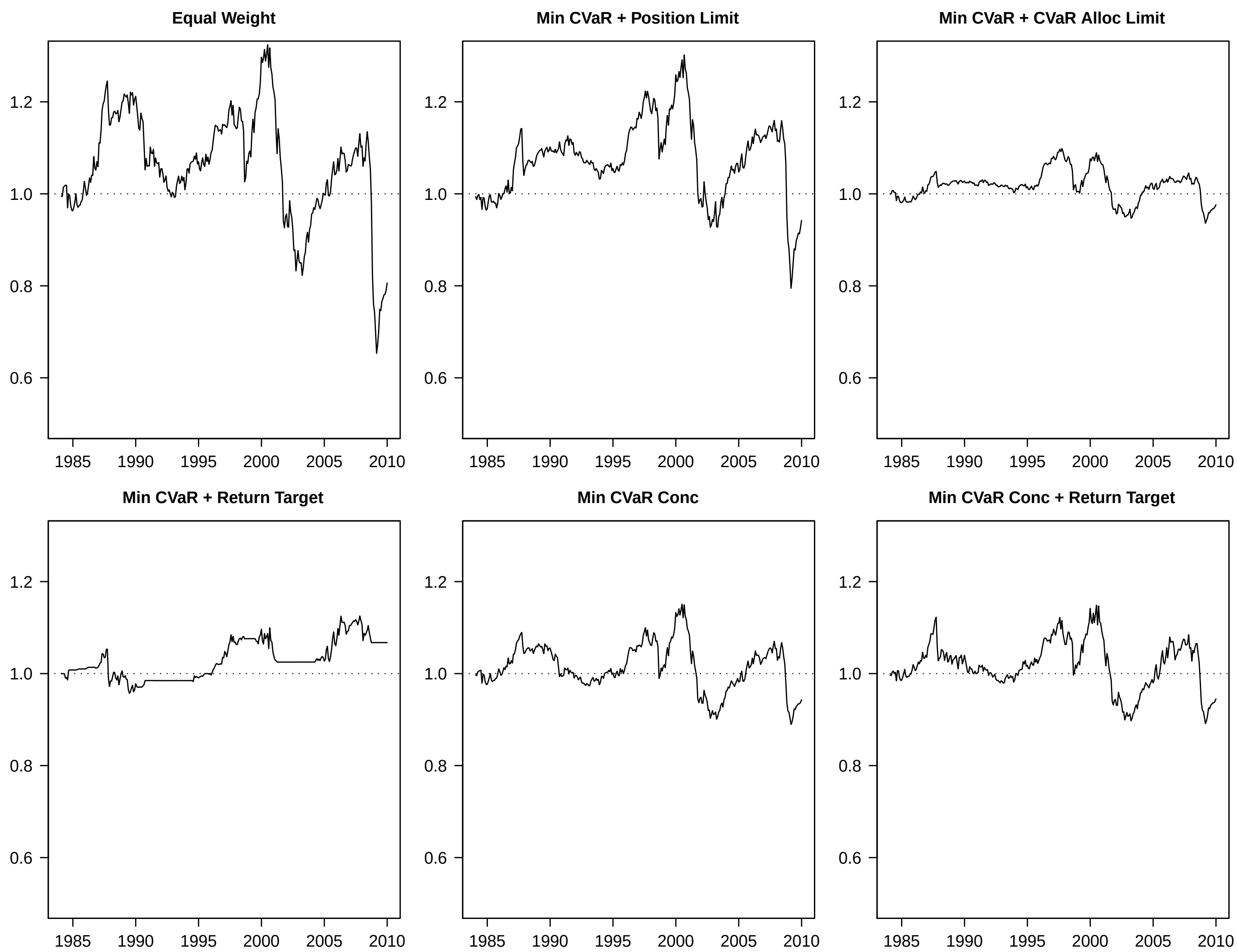


## Out of sample performance:

| | Equal | Min CVaR | | | | Min CVaR Conc | |
|---|---|---|---|---|---|---|---|
| | Weight | | Position Limit | CVaR Alloc Limit | Return Target | | Return Target |
| Mean (in %) | 0.40 | 0.44 | 0.43 | 0.43 | 0.47 | 0.43 | 0.43 |
| StdDev (in %) | 2.85 | 1.38 | 2.41 | 1.64 | 1.74 | 1.85 | 2.02 |
| Hist 95% CVaR (in %) | 6.95 | 2.72 | 5.78 | 3.49 | 3.59 | 4.11 | 4.52 |
| Portfolio turnover (in %) | 1.27 | 2.30 | 3.10 | 2.70 | 4.50 | 1.82 | 4.45 |

$$\text{Portf. turnover} = \frac{1}{NT_*} \sum_{t=1}^{T_*-1} \left| w_{(i)t+1} - w_{(i)t+} \right| .$$

# Out of sample cum performance, relative to min CVaR:



## Drawdowns higher than 10% on portfolio strategies over the period January 1984-December 2009:

| | Equal Weight | Min CVaR | | | | Min CVaR Conc | |
|---|---|---|---|---|---|---|---|
| | | | Position Limit | CVaR Alloc Limit | Return Target | | Return Target |
| Credit crisis* | 0.47 | 0.10 | 0.37 | 0.18 | 0.14 | 0.24 | 0.24 |
| Dot-com bubble burst** | 0.28 | | 0.19 | | | 0.11 | 0.11 |
| Asian-Russian crisis*** | 0.13 | | 0.12 | | | | 0.11 |
| Black Monday**** | 0.11 | | 0.13 | | 0.12 | | 0.12 |

* Dec 2007-Oct 2008 for the Min CVaR strategy, Nov 2007-Feb 2009 for all other styles.

** Start: Sept 2000 for all styles. End: Jan 2002 for Min CVaR Conc styles, July 2002 for the Min CVaR with position limit style, September 2002 for all other styles.

*** Aug 1997-Aug 1998 for equal-weight strategy, Oct 1997-Aug 1998 for the Min CVaR + Return target and Nov 1997-Aug 1998 for the Min CVaR with position limit style.

**** Black Monday: Sept-November 1987.

# Conclusion

CVaR budgets are useful for:

- Ex post analysis of the portfolio risk allocation;
- **And** input in the portfolio allocation strategy through
  - ✓ minimum CVaR Concentration objective
  - ✓ and/or risk allocation constraints.

# References

- Software: R packages DEoptim, PerformanceAnalytics and PortfolioAnalytics
- Related research papers:


1. With B. Peterson and C. Croux: Estimation and decomposition of downside risk for portfolios with non-normal returns. Journal of Risk, Winter 2008.
2. With B. Peterson: Component VaR for a non-normal world. RISK, November 2008.
3. With D. Ardia, P. Carl, K. Mullen and B. Peterson. DEoptim for non-convex portfolio optimization. SSRN.
4. With P. Carl and B. Peterson. Portfolio optimization with CVaR budgets.

# Appendix

## $\%CVaR_{(1)} = f(w_{(1)})$ for bivariate normal portfolio: