Portfolio optimization with CVaR budgets

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Risk budgets:

- Standard tool to quantify risk allocation;
- Previous research: non-normality return series, CVaR: PerformanceAnalytics.
- Instrument to adjust marginally portfolios;
- This research: Use of risk budgets as objective and/or constraint in portfolio allocation styles: PortfolioAnalytics;
- Collaborative: Peter Carl & Brian Peterson, David Ardia, Christophe Croux.
Bender, Briand, Nielsen, Stefek (JPM, Winter 2010):

“Traditional approaches to structuring policy portfolios for strategic asset allocation have not provided the full potential of diversification. Portfolios based on a 60/40 allocation between equities and bonds remain volatile and dominated by equity risk.”

Minimum risk portfolios tend to be dominated by bond risk and have lower expected returns.
Bender, Briand, Nielsen, Stefek (JPM, Winter 2010):

“Traditional approaches to structuring policy portfolios for strategic asset allocation have not provided the full potential of diversification. Portfolios based on a 60/40 allocation between equities and bonds remain volatile and dominated by equity risk.”

Minimum risk portfolios tend to be dominated by bond risk and have lower expected returns.

Optimize the risk allocation directly in the portfolio strategy. Examples:

✓ A 60/40 risk allocation portfolio or an equal-risk portfolio

✓ The most risk diversified portfolio subject to return/risk targets.
Risk budgets:

- Review on risk budgets;
- Use of risk budgets as objective and/or constraint in portfolio allocation styles.
- Illustrations:
  - Static bond-equity portfolio, R code (DEoptim, see also Guy Yollin’s slides RFinance 2009);
  - Dynamic 4 assets.
A primer on risk budgets
VaR budget on 60/40 portfolio

> library(PortfolioAnalytics)
> data(indexes)
> head(indexes[,1:2])

<table>
<thead>
<tr>
<th>Date</th>
<th>US Bonds</th>
<th>US Equities</th>
</tr>
</thead>
<tbody>
<tr>
<td>1980-01-31</td>
<td>-0.0271</td>
<td>0.0610</td>
</tr>
<tr>
<td>1980-02-29</td>
<td>-0.0669</td>
<td>0.0031</td>
</tr>
<tr>
<td>1980-03-31</td>
<td>0.0053</td>
<td>-0.0987</td>
</tr>
<tr>
<td>1980-04-30</td>
<td>0.0992</td>
<td>0.0429</td>
</tr>
<tr>
<td>1980-05-31</td>
<td>0.0000</td>
<td>0.0562</td>
</tr>
<tr>
<td>1980-06-30</td>
<td>0.0605</td>
<td>0.0296</td>
</tr>
</tbody>
</table>
VaR budget on 60/40 portfolio

<table>
<thead>
<tr>
<th></th>
<th>US Bonds</th>
<th>US Equities</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.006916</td>
<td>0.008238</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>0.018101</td>
<td>0.044766</td>
</tr>
</tbody>
</table>

```r
> apply(indexes[,1:2],2,'mean')
  US Bonds US Equities
  0.006916187  0.008238420

> apply(indexes[,1:2],2,'sd')
  US Bonds US Equities
  0.01810161  0.04476569
```
VaR budget on 60/40 portfolio

> w6040 <- c(0.4, 0.6)
> library(PerformanceAnalytics)
> VaR(R=indexes[,1:2], weights=w6040,
+     portfolio_method="component")

$MVaR

[1,] 0.04336715

$contribution

       US Bonds  US Equities
-0.0002303964  0.0435975440

$pct_contrib_MVaR

       US Bonds  US Equities
-0.005312695  1.005312695
\[ C_i \text{VaR} = w_i \frac{\partial \text{VaR}(w)}{\partial w_i} \]

Gouriéroux, Laurent and Scaillet (2000):

\[ C_i \text{VaR} = -E[w_ir_ir_p = -\text{VaR}] \]

Estimation:

✓ Simulation

✓ Explicit formulae Cornish-Fisher estimator (Boudt, Peterson and Croux, 2008; Peterson and Boudt, 2008).
Pearson [2002, p.7]: “Value-at-risk has some well known limitations, and it may be that some other risk measures eventually supplants value-at-risk in the risk budgeting process.”

CVaR:

- Coherent risk measure (most notably: subadditive);
- Less incentive to load on the tail risk below the VaR used.
CVaR budgets

Outline
A primer on risk budgets
VaR budget
CVaR budget
Min CVaR portfolio
CVaR budgets as objective or constraint in portfolio allocation
Dynamic portfolio allocation
Conclusion
Appendix

\[ C_i \text{CVaR} = w_i \frac{\partial \text{CVaR}(w)}{\partial w_i} \]

- Scaillet (2002):

\[ C_i \text{CVaR} = -E[w_i r_i | r_p \leq -\text{VaR}] \]

- Estimation:

  ✓ Simulation

  ✓ Explicit formulae Cornish-Fisher estimator (Boudt, Peterson and Croux, 2008).
CVaR budgets

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> ES(R=indexes[,1:2], weights=w6040, + portfolio_method="component")

$MES
[1,] 0.07725177

$contribution

     US Bonds   US Equities
-0.001066194  0.078317964

$pct_contrib_MES

     US Bonds   US Equities
-0.01380155  1.01380155
CVaR in portfolio allocation

- As an objective: Minimum CVaR portfolio (E.g. Rockafellar and Uryasev, 2000)

- As a constraint: Min CVaR/SD portfolio under CVaR constraints (E.g. Alexander and Baptista, 2004, Krokhmal, Palmquist and Uryasev, 2002).

> library(DEoptim)
> obj <- function(w) {
+   if (sum(w) == 0) { w <- w + 1e-2 }
+   w <- w / sum(w)
+   ES(R=indexes[,1:2],weights = w)$MES
+ }
> out <- DEoptim(fn = obj, lower = rep(0, 2),
+                upper = rep(1, 2), DEoptim.control(itermax=50))
> wstar <- out$optim$bestmem
> wMinCVaR <- wstar / sum(wstar)
> print(wMinCVaR)

US Bonds          US Equities
0.96443348        0.03556652
```r
> ES(R=indexes[,1:2], weights=wMinCVaR, + portfolio_method="component")

$MES

```

```
[1,] 0.01102894
```

$contribution

```
US Bonds  US Equities
0.0106366796  0.0003922610
```

$pct_contrib_MES

```
US Bonds US Equities
0.96443349  0.03556651
```
### CVaR budgets

<table>
<thead>
<tr>
<th>style</th>
<th>Weight allocation</th>
<th>Risk allocation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>bond</td>
<td>equity</td>
</tr>
<tr>
<td>60/40 weight</td>
<td>0.40</td>
<td>0.6</td>
</tr>
<tr>
<td>60/40 risk alloc</td>
<td>0.84</td>
<td>0.16</td>
</tr>
<tr>
<td>Min CVaR Conc</td>
<td>0.86</td>
<td>0.14</td>
</tr>
<tr>
<td>Min CVaR</td>
<td>0.96</td>
<td>0.04</td>
</tr>
</tbody>
</table>
CVaR budgets as objective or constraint in portfolio allocation
Traditional use

Risk contribution $C_i \text{CVaR}(w) = w_i \frac{\partial \text{CVaR}(w)}{\partial w_i}$


Keel and Ardia (2009):

1. Only precise for infinitesimal changes, poor approximations for realistic reallocations.
2. Assume changing a single position keeping fixed all other positions $\gg$ full investment constraint.
Proposal I: Risk budget constraints

- Avoid downside risk concentration by:

\[ l_i \leq \%C_i \cdot \text{CVaR} \equiv \frac{C_i \cdot \text{CVaR}}{\text{CVaR}} \leq u_i \]

- Min CVaR portfolio with
  
  - 60/40 risk allocation constraint
  - equal risk allocation constraint.
> obj <- function(w) {
+ if (sum(w) == 0) { w <- w + 1e-2 }
+ w <- w / sum(w)
+ CVaR <- ES(R=indexes[,1:2],weights = w)
+ tmp1 <- CVaR$MES
+ tmp2 <- max(CVaR$pct_contrib_ES
+ - c(0.405, 0.605) , 0)
+ tmp3 <- max(c(0.395, 0.595) -
+ CVaR$pct_contrib_ES , 0)
+ out <- tmp1 + 1e3 * tmp2 + 1e3 * tmp3
+ }

60/40 risk allocation portfolio
> out <- DEoptim(fn = obj, lower = rep(0, 2),
+       upper = rep(1, 2), DEoptim.control(itermax=50))
> wstar <- out$optim$bestmem
> w6040riskalloc <- wstar / sum(wstar)
> print(w6040riskalloc)

      US Bonds  US Equities
US Bonds   0.8382035  0.1617965
> ES(R=indexes[,1:2], weights=w6040riskalloc, + portfolio_method="component")

$MES
[1,] 0.01400341

$contribution

US Bonds US Equities
0.005671224 0.008332185

$pct_contrad_MES

US Bonds US Equities
0.4049888 0.5950112
Special case: Equal-risk portfolio

- Min CVaR with equal-risk constraint

\[ \%C_i \text{CVaR}(w) = \frac{1}{N} \quad (i = 1, \ldots, N) \]

- In this portfolio:

\[ \frac{w_i}{w_j} = \frac{\partial \text{CVaR}/\partial w_j}{\partial \text{CVaR}/\partial w_i} \]

Downweights “hot spots”: positions for which a marginal decrease in weight leads to a large reduction in CVaR.
Avoid downside risk concentration by:

$$\min_{w} \max_{i} C_i \text{CVaR}(w)$$
Proposal II: Risk budget objective

- Avoid downside risk concentration by:

\[
\min_{w} \max_{i} C_i \text{CVaR}(w)
\]

- Objective trades off Risk Minimization and Risk Diversification, since:

\[
\max_{i} C_i \text{CVaR} = \text{CVaR} \max\{\%C_1 \text{CVaR}, \ldots, \%C_N \text{CVaR}\}
\]
If the set of equal-risk portfolios is non-empty and the minimum CVaR concentration portfolio has a unique optimum. Then the minimum CVaR concentration portfolio criterion is equivalent to:

\[
\min_w \text{CVaR}(w)
\]

subject to \(\%C_1 \text{CVaR} = \ldots = \%C_N \text{CVaR}\)

But computationally more simple and has also a solution if there is no equal-risk portfolio.
Min CVaR Concentration portfolio

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```r
> obj <- function(w) {
+   if (sum(w) == 0) { w <- w + 1e-2 }
+   w <- w / sum(w)
+   CVaR <- ES(R=indexes[,1:2],weights = w)
+   out <- max(CVaR$contribution)
+ }

> out <- DEoptim(fn = obj, lower = rep(0, 2),
+                upper = rep(1, 2), DEoptim.control(itermax=50))
> wstar <- out$optim$bestmem
> wMinCVaRConc <- wstar / sum(wstar)
> print(wMinCVaRConc)

US Bonds US Equities
0.8584465 0.1415535
```
Min CVaR Concentration portfolio

> ES(R=indexes[,1:2], weights=wMinCVaRConc, + portfolio_method="component")

$MES

[1,] 0.01315665

$contribution

US Bonds US Equities
0.006578325 0.006578323

$pct_contrib_MES

US Bonds US Equities
0.5000001 0.4999999
### Overview portfolio allocations

<table>
<thead>
<tr>
<th>style</th>
<th>Weight allocation</th>
<th>Risk allocation</th>
</tr>
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<tbody>
<tr>
<td></td>
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Adding a return target: Efficient frontier

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Dynamic portfolio allocation
We consider the following investment strategies, quarterly rebalancing, 4 assets:

1. Benchmark strategies:
   - “Equal Weight”
   - “Min CVaR”
   - “Min CVaR + 40% Position Limit”
   - “Min CVaR + EW return target”

2. Strategies that use CVaR budgets:
   - “Min CVaR + 40% Perc CVaR Alloc Limit”
   - “Min CVaR Conc”
   - “Min CVaR Conc + EW return target”
### Summary statistics data:

4 assets: Merrill Lynch US bond, S&P 500, MSCI EAFE and S&P GSCI

Real monthly returns Jan 1976-December 2009, total return indices

<table>
<thead>
<tr>
<th></th>
<th>US bond</th>
<th>S&amp;P 500</th>
<th>MSCI EAFE</th>
<th>S&amp;P GSCI</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean (in %)</td>
<td>0.32</td>
<td>0.52</td>
<td>0.39</td>
<td>0.10</td>
</tr>
<tr>
<td>StdDev (in %)</td>
<td>1.86</td>
<td>4.46</td>
<td>4.98</td>
<td>5.50</td>
</tr>
<tr>
<td>Historical 95% CVaR (in %)</td>
<td>3.64</td>
<td>10.64</td>
<td>12.46</td>
<td>13.58</td>
</tr>
</tbody>
</table>

Quarterly rebalanced based on time-varying conditional moment estimates (EWMA mean, GARCH volatility, rolling 8 year correlation, coskewness and cokurtosis).
Weight allocation:

- Min CVaR
- Min CVaR + CVaR Alloc Limit
- Min CVaR + Return Target
- Min CVaR Conc
- Equal Weight
- Min CVaR Conc + Return Target

Legend:
- US bond
- S&P 500
- EAFE
- GSCI
CVaR allocation:

- Min CVaR
- Min CVaR + CVaR Alloc Limit
- Min CVaR + Return Target
- Min CVaR Conc
- Min CVaR Conc + Return Target
- Equal Weight

Legend:
- US bond
- S&P 500
- EAFE
- GSCI
### Out of sample performance:

<table>
<thead>
<tr>
<th></th>
<th>Equal Weight</th>
<th>Min CVaR Position Limit</th>
<th>CVaR Alloc Limit</th>
<th>Return Target</th>
<th>Min CVaR Conc Return Target</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean (in %)</td>
<td>0.40</td>
<td>0.44</td>
<td>0.43</td>
<td>0.43</td>
<td>0.47</td>
</tr>
<tr>
<td>StdDev (in %)</td>
<td>2.85</td>
<td>1.38</td>
<td>2.41</td>
<td>1.64</td>
<td>1.74</td>
</tr>
<tr>
<td>Hist 95% CVaR (in %)</td>
<td>6.95</td>
<td>2.72</td>
<td>5.78</td>
<td>3.49</td>
<td>3.59</td>
</tr>
<tr>
<td>Portfolio turnover (in %)</td>
<td>1.27</td>
<td>2.30</td>
<td>3.10</td>
<td>2.70</td>
<td>4.50</td>
</tr>
</tbody>
</table>

Portf. turnover = \( \frac{1}{NT^*} \sum_{t=1}^{T^*-1} |w(i)_{t+1} - w(i)_t| \).
Out of sample cum performance, relative to min CVaR:

- Equal Weight
- Min CVaR + Position Limit
- Min CVaR + CVaR Alloc Limit
- Min CVaR + Return Target
- Min CVaR Conc
- Min CVaR Conc + Return Target
Drawdowns higher than 10% on portfolio strategies over the period January 1984-December 2009:

<table>
<thead>
<tr>
<th>Event</th>
<th>Equal Weight</th>
<th>Min CVaR Weight</th>
<th>Min CVaR Position Limit</th>
<th>Min CVaR CVaR Alloc Limit</th>
<th>Min CVaR Return Target</th>
<th>Min CVaR Conc Return Target</th>
</tr>
</thead>
<tbody>
<tr>
<td>Credit crisis</td>
<td>0.47</td>
<td>0.10</td>
<td>0.37</td>
<td>0.18</td>
<td>0.14</td>
<td>0.24</td>
</tr>
<tr>
<td>Dot-com bubble burst</td>
<td>0.28</td>
<td>0.19</td>
<td></td>
<td></td>
<td></td>
<td>0.11</td>
</tr>
<tr>
<td>Asian-Russian crisis</td>
<td>0.13</td>
<td>0.12</td>
<td></td>
<td></td>
<td></td>
<td>0.11</td>
</tr>
<tr>
<td>Black Monday</td>
<td>0.11</td>
<td>0.13</td>
<td></td>
<td></td>
<td>0.12</td>
<td>0.12</td>
</tr>
</tbody>
</table>


** Start: Sept 2000 for all styles. End: Jan 2002 for Min CVaR Conc styles, July 2002 for the Min CVaR with position limit style, September 2002 for all other styles.


Conclusion
CVaR budgets are useful for:

- Ex post analysis of the portfolio risk allocation;
- And input in the portfolio allocation strategy through
  - minimum CVaR Concentration objective
  - and/or risk allocation constraints.
Software: R packages DEoptim, PerformanceAnalytics and PortfolioAnalytics

Related research papers:
\% CVaR(1) = f(w(1)) for bivariate normal portfolio: