R in Finance

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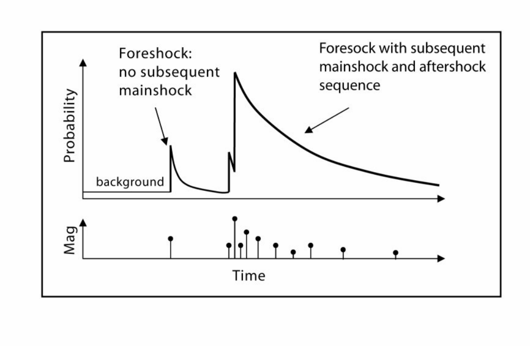
Mutually exciting Hawkes processes for the microstructure noise modeling

Clément Dunand-Châtellet (PhD student)

Ecole Polytechnique, Palaiseau, France

Introduction

From Physics to finance ...



Properties of earthquakes

- **Irregularly** spaced in time, **clustering** of erthquakes
- **Stochastic** magnitude of main shock
- **Correlations** between **main shock and aftershocks**: Omori's Law
- **Correlation in space** between different areas of the Earth crust

Modeling earthquakes with Hawkes processes

- **Point process with conditional intensity**
- Background activity
- **Correlations between areas A and B**

Hawkes processes

$$\lambda_A(t|Ht) = \mu + \sum_{t<t_i} \phi(t - t_i) + \sum_{t<t_j} \psi(t - t_j) + Periodic \quad functions \tag{1}$$

- λ: Conditional intensity
- μ: Background activity
- ϕ: Response function of a shock from region A at time t_i
- ψ: Response function of a shock from region B at time t_j

Conditional intensity

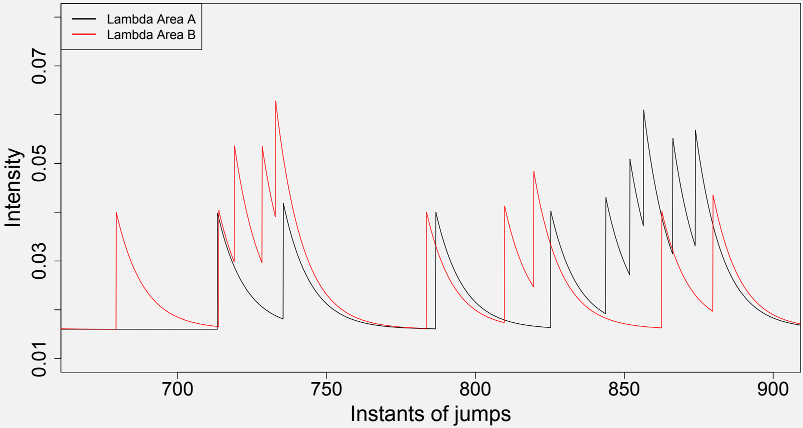


FIGURE 1: Conditional intensity

Some similarities ?

High Frequency data

- **Irregular** spacing of data in time, clustering of orders
  Irregular spacing of earthquakes
- Discrete nature of price variations **tick data**
  Discrete variations of magnitudes
- **Autocorrelation** of assets
  Main Shock and aftershocks
- **Cross correlation** between assets
  Regions A and B

$\Rightarrow$ *Prices are point processes living on a tick grid*

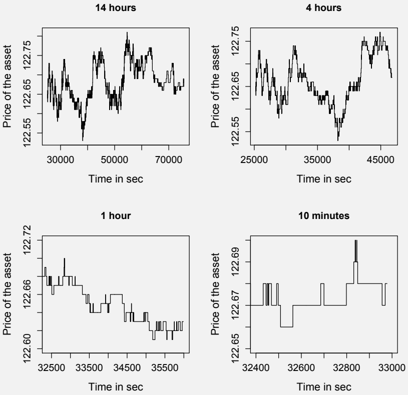


Figure 2: Discrete nature of prices of FBund the 07th of december 2009 (maturity 03/2010)

Objectives

- Estimate **realized volatility** at high frequency
- Estimate **correlations** between assets at high frequency
- Design a **cross scale** model

Constraints

- Realized **volatility is not stable** at fine scale
- Strong **mean reversion** at very small scales
- **Correlations** between assets **vanish** at small scales

Tools

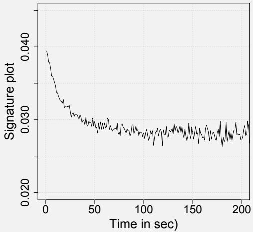
- Volatility: **Signature plot**
- Mean reversion: histograms
- Correlations: **Epps effect**

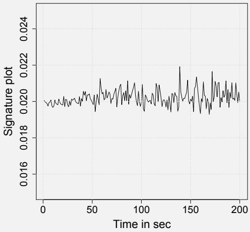
First stylized fact: Signature plot (Andersen [2])

- **Definition:** the **realized volatility** over a period $[0, T]$ at scale τ

$X(t)$: price of the asset at time t (last traded, mid price, ...)

$$\hat{V}(\tau) = \frac{1}{T} \sum_{n=0}^{T/\tau} (X((n+1)\tau) - X(n\tau))^2 \quad (2)$$





SECOND STYLIZED FACT: EPPS EFFECT (EPPS [3])

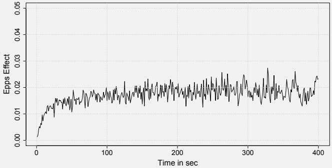
- **Definition:** A **correlation coefficient** over a time period $[0, T]$ of price increments of two assets

$X_1(t)$ and $X_2(t)$: prices of two assets at time t

$$\hat{\rho}(\tau) = \frac{\hat{V_{12}}(\tau)}{\sqrt{\hat{V_1}(\tau)\hat{V_2}(\tau)}} \tag{3}$$

where

$$\hat{V_{12}}(\tau) = \frac{1}{T}\sum_{n=0}^{T/\tau}[X_1((n+1)\tau) - X_1(n\tau)][X_2((n+1)\tau) - X_2(n\tau)] \tag{4}$$



MODEL

A "FINE TO COARSE" MODEL [1]

Defined at a microscopic scale (tick-by-tick data modeling by means of marked point processes with appropriate stochastic intensities)

$\Rightarrow$ 4 key points in the model :

1. Diffuse as a **brownian diffusion in large scale** (cross scale model)
2. Incorporate **microstructure noise** (mean reversion) through coupled **Hawkes processes**
3. Restriction of parameters to ensure stationarity, non-negativity and stability
4. Account for **signature plot and Epps effect**

The mutually excited price model (MEP) in 1D

$$X(t) = N^{+}(t) - N^{-}(t) \tag{5}$$

where $N^{\pm}(t)$ are Hawkes processes with random intensities $\lambda^{\pm}(t)$ given by :

$$\lambda^{\pm}(t) = \frac{\mu}{2} + \int_0^t \phi(t-s)dN^{\mp}(s) \tag{6}$$

where μ is an exogeneous intensity and $\phi(t)$ a mutually exciting kernel

Stability conditions

- $\phi(t) > 0$ and $Supp(\phi) \subset \mathbb{R}_+$ (To ensure non-negativity)
- Stability $\equiv ||\phi||_1$ (X(t) has stationary increments)
- Particular case $\phi(t) = \alpha e^{-\beta t}\mathbf{1}_{\mathbb{R}_+}(t)$ ($||\phi||_1 = \frac{\alpha}{\beta} < 1$ To ensure stationarity and stability)
  - $\alpha > 0$ is a scale parameter
  - $\beta > 0$ drives the strength of the time decay

Conditional intensity

The more $X(t)$ goes up, the greater the intensity of λ_-
The more $X(t)$ goes down, the greater the intensity of λ_+

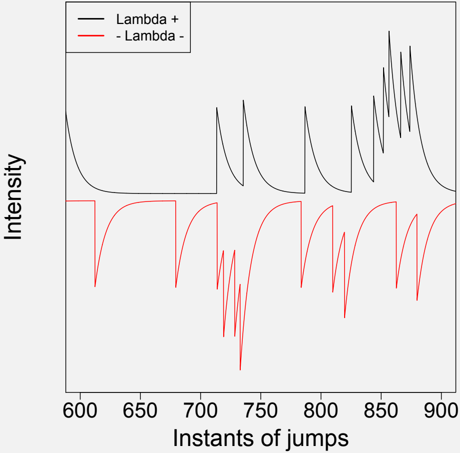


FIGURE 3: Intensities of λ^+ and $-\lambda^-$

Closed form formula for the mean signature plot of the MEP model

- $X(t)$ on $[0, T]$ with initial condition $X(0) = 0$

$$V(\tau) = \frac{1}{T} \sum_{n=0}^{T/\tau} (X(n\tau + \tau) - X(n\tau))^2 \tag{7}$$

which can be written as :

$$V(\tau) = \mathbb{E}[\hat{V}(\tau)] = \Lambda \left[\psi^2 + (1 - \psi^2) \frac{1 - e^{-\gamma\tau}}{\gamma\tau} \right] \tag{8}$$

where $\Lambda = \frac{2\mu}{1-||\phi||_1}$, $\quad \psi = \frac{1}{1+||\phi||_1}$, $\quad$ *and* $\quad \gamma = \alpha + \beta$

- Closed Form formula for the mean Epps effect is too long to be exposed here, but can be computed !

The mutually excited price model (MEP) in 2D

$$X(t) = N_1(t) - N_2(t) \qquad and \qquad Y(t) = N_3(t) - N_4(t) \tag{9}$$

where $N_i(t)$ are Hawkes processes with random intensities $\lambda_i(t)$ given by :

$$\lambda_X^{\pm}(t) = \frac{\mu_X^{\pm}}{2} + \int_0^t \phi_{X,X}(t-s)dN_X^{\mp}(s) + \int_0^t \phi_{X,Y}(t-s)dN_Y^{\pm}(s) \tag{10}$$

and

$$\lambda_Y^{\pm}(t) = \frac{\mu_Y^{\pm}}{2} + \int_0^t \phi_{Y,Y}(t-s)dN_Y^{\mp}(s) + \int_0^t \phi_{Y,X}(t-s)dN_X^{\pm}(s) \tag{11}$$

Numerical Validation

One Hawkes' process

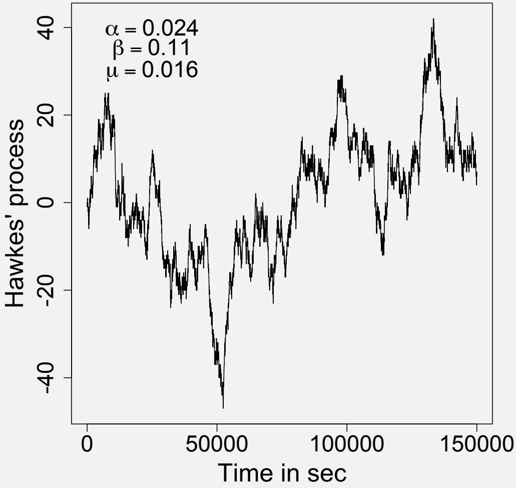


FIGURE 4: Simulation of a Hawkes' process for $\alpha = 0.024$, $\beta = 0.11$ and $\mu = 0.016$

Associated signature plot to Hawkes process

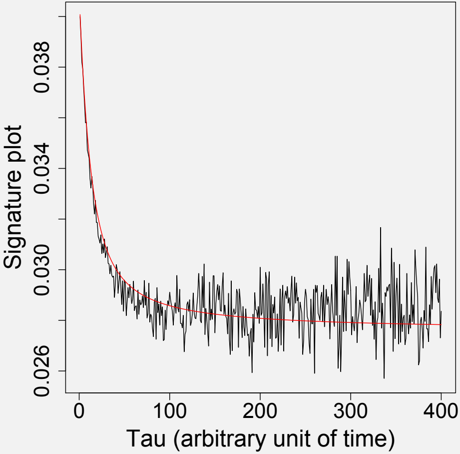


FIGURE 5: Signature Plot computed in link with figure 4

$$\hat{\theta_{reg}} = Argmin_{\theta}|\hat{V}(\tau) - V(\tau)|^2 \quad (12)$$

- constraints: $\mu > 0, \alpha > 0, \beta > 0$ and $\frac{\alpha}{\beta} < 1$
- $\hat{\alpha} = 0.024$
- $\hat{\beta} = 0.11$
- $\hat{\mu} = 0.016$

Several Hakwes processes in 2D

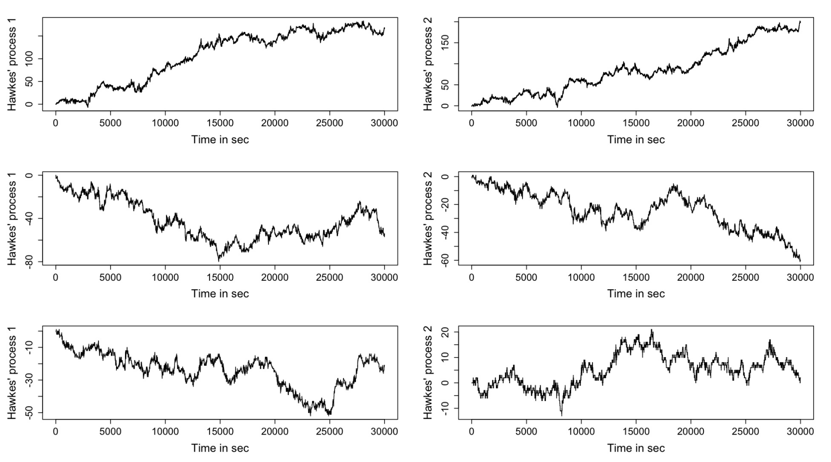


FIGURE 6: Simulation of a Hawkes' process for different parameters

Associated Epps effect to Hawkes processes

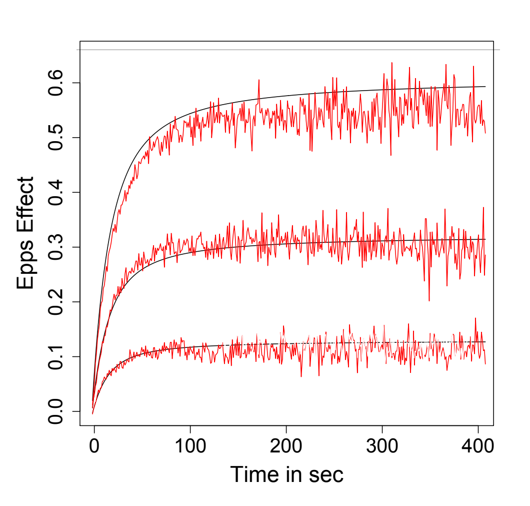


FIGURE 7: Empirical and theoretical Epps effect

Real Life Example

Definition of assets and assumptions

- **Euro-Bund futures** contracts or **Euro-Bobl futures** contracts
- Tick-by-tick (0.01) **last traded price**
- Trading time between 8 AM and 10 PM (intraday seasonality)

- **Assumption 1:** parameters of the model are **piecewise constant** in the intraday regime and the distribution of parameters is also constant for every day

$\Rightarrow$ **stationarity** and restriction to the time period 9 AM to 11 AM from 01/11/2009 to 15/12/2009 (21 days) on the contract maturity 12/2009

- **Assumption 2:** Only jumps of size $\pm = 1$

$\Rightarrow$ Price dynamics is marginally **modified**

Mean reverting behavior

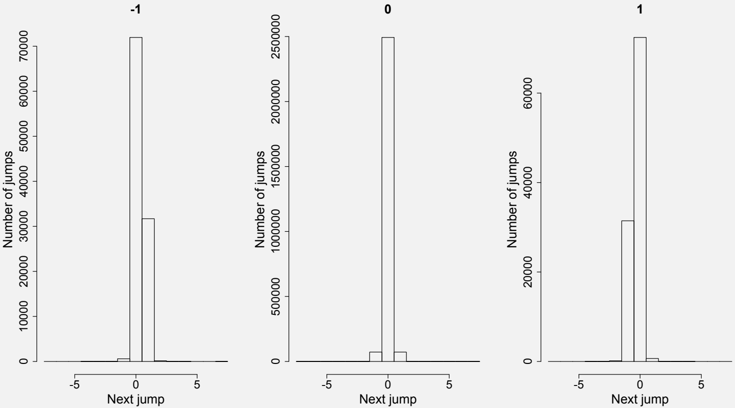


FIGURE 8: Histograms of jumps frequencies following a given type of jump, revealing a **mean reverting behavior**

Conditional jumps

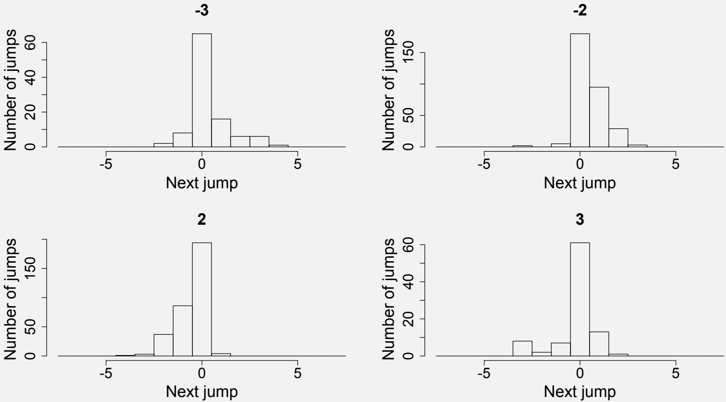


FIGURE 9: Histograms of jumps following the jumps with an amplitude greater than 1

Signature plot over 21 days

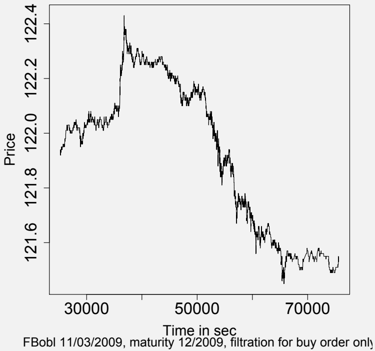


FIGURE 10: Price of the Euro-Bobl during the whole day 11/03/2009 (maturity 12/2009)

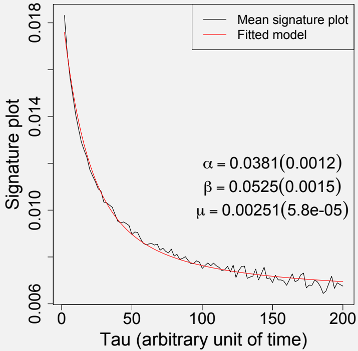


FIGURE 11: Empirical mean signature plot computed for november's opening day **for buy orders with jumps of size** ± 1

Euro-Bund Euro-Bobl prices

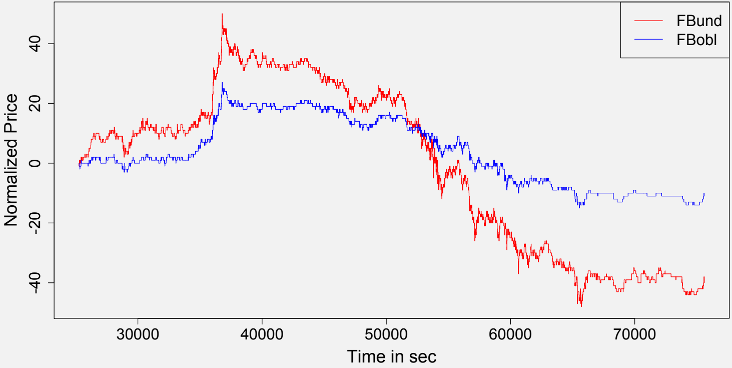


FIGURE 12: Last traded price prices path of the Euro-Bobl and Euro-Bund contract on the 3rd of november 2009 (maturity 12/2009)

Self and cross correlation

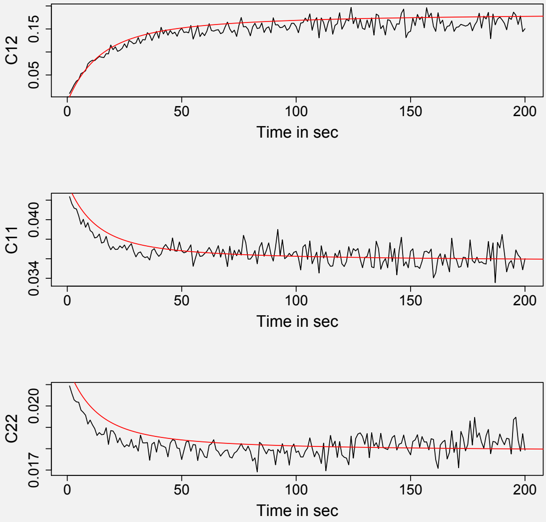


FIGURE 13: Self and cross correlations between Euro-Bobl and Euro-Bund

Conclusion and Outlooks

Advantages

- Simple tick-by-tick price model based on (mutually exciting) **Hawkes processes**
- **Closed form** expressions for second order properties at all time scales
- Ability to recover major high frequency stylized facts (**Signature plot and Epps effect**)

$\Rightarrow$ Simple tool to investigate intraday market features

Downsides

- Constant parameters μ, α and β
- Large Number of parameters when couplings assets
- **No volume**

Outlooks

- **Implementation of a high frequency trading strategy**
- Backtest

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CONTACT

- Clément Dunand-Châtellet
- **cdunandc@gmail.com**

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