**R in Finance**

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**Mutually exciting Hawkes processes for the microstructure noise modeling**

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1. **Introduction**

2. **Model**

3. **Numerical Validation**

4. **Real Life Example**

5. **Conclusion and Outlooks**
**Properties of Earthquakes**

- **Irregularly** spaced in time, *clustering* of earthquakes
- **Stochastic** magnitude of main shock
- **Correlations** between main shock and aftershocks: Omori’s Law
- **Correlation in space** between different areas of the Earth crust
Earthquake’s modeling

**MODELING EARTHQUAKES WITH HAWKES PROCESSES**

- **Point process with conditional intensity**
- **Background activity**
- **Correlations between areas A and B**

**HAWKES PROCESSES**

\[
\lambda_A(t|Ht) = \mu + \sum_{t<t_i} \phi(t - t_i) + \sum_{t<t_j} \psi(t - t_j) + \text{Periodic functions} \tag{1}
\]

- \(\lambda\): Conditional intensity
- \(\mu\): Background activity
- \(\phi\): Response function of a shock from region A at time \(t_i\)
- \(\psi\): Response function of a shock from region B at time \(t_j\)
**Figure 1:** Conditional intensity
Some similarities?

**High Frequency data**

- **Irregular** spacing of data in time, clustering of orders
  *Irregular spacing of earthquakes*
- Discrete nature of price variations **tick data**
  *Discrete variations of magnitudes*
- **Autocorrelation** of assets
  *Main Shock and aftershocks*
- **Cross correlation** between assets
  *Regions A and B*

⇒ *Prices are point processes living on a tick grid*

**Figure 2:** Discrete nature of prices of FBund the 07th of December 2009 (maturity 03/2010)
**OBJECTIVES**
- Estimate **realized volatility** at high frequency
- Estimate **correlations** between assets at high frequency
- Design a **cross scale** model

**CONSTRAINTS**
- Realized **volatility is not stable** at fine scale
- Strong **mean reversion** at very small scales
- **Correlations** between assets **vanish** at small scales

**TOOLS**
- Volatility: **Signature plot**
- Mean reversion: histograms
- Correlations: **Epps effect**
**First stylized fact: Signature plot (Andersen [2])**

- **Definition:** the realized volatility over a period \([0, T]\) at scale \(\tau\)

\[ \hat{V}(\tau) = \frac{1}{T} \sum_{n=0}^{T/\tau} (X((n+1)\tau) - X(n\tau))^2 \]  \hfill (2)

\(X(t)\): price of the asset at time \(t\) (last traded, mid price, ...)

![Signature plot](image1)

![Signature plot](image2)
SECOND STYLIZED FACT: EPPS EFFECT (EPPS [3])

- **Definition:** A correlation coefficient over a time period $[0, T]$ of price increments of two assets

$X_1(t)$ and $X_2(t)$: prices of two assets at time $t$

$$
\hat{\rho}(\tau) = \frac{\hat{V}_{12}(\tau)}{\sqrt{\hat{V}_1(\tau)\hat{V}_2(\tau)}}
$$

(3)

where

$$
\hat{V}_{12}(\tau) = \frac{1}{T} \sum_{n=0}^{T/\tau} [X_1((n + 1)\tau) - X_1(n\tau)][X_2((n + 1)\tau) - X_2(n\tau)]
$$

(4)
MODEL
A ”FINE TO COARSE” MODEL [1]

Defined at a microscopic scale (tick-by-tick data modeling by means of marked point processes with appropriate stochastic intensities)

⇒ 4 key points in the model :

1. Diffuse as a brownian diffusion in large scale (cross scale model)
2. Incorporate microstructure noise (mean reversion) through coupled Hawkes processes
3. Restriction of parameters to ensure stationarity, non-negativity and stability
4. Account for signature plot and Epps effect
The mutually excited price model (MEP) in 1D

\[ X(t) = N^+(t) - N^-(t) \]  \hspace{1cm} (5)

where \( N^\pm(t) \) are Hawkes processes with random intensities \( \lambda^\pm(t) \) given by:

\[ \lambda^\pm(t) = \frac{\mu}{2} + \int_0^t \phi(t-s)dN^\mp(s) \]  \hspace{1cm} (6)

where \( \mu \) is an exogeneous intensity and \( \phi(t) \) a mutually exciting kernel.

Stability conditions

- \( \phi(t) > 0 \) and \( Supp(\phi) \subset \mathbb{R}_+ \) (To ensure non-negativity)
- Stability \( \equiv ||\phi||_1 \) (X(t) has stationary increments)
- Particular case \( \phi(t) = \alpha e^{-\beta t}1_{\mathbb{R}_+}(t) \) (\( ||\phi||_1 = \frac{\alpha}{\beta} < 1 \) To ensure stationarity and stability)
  - \( \alpha > 0 \) is a scale parameter
  - \( \beta > 0 \) drives the strength of the time decay
The more $X(t)$ goes up, the greater the intensity of $\lambda_-$

The more $X(t)$ goes down, the greater the intensity of $\lambda_+$

**Figure 3:** Intensities of $\lambda^+$ and $-\lambda^-$
Closed Form Formula for the Mean Signature Plot of the MEP Model

- $X(t)$ on $[0, T]$ with initial condition $X(0) = 0$

$$V(\tau) = \frac{1}{T} \sum_{n=0}^{T/\tau} (X(n\tau + \tau) - X(n\tau))^2$$

which can be written as:

$$V(\tau) = \mathbb{E}[\hat{V}(\tau)] = \Lambda \left[ \psi^2 + (1 - \psi^2) \frac{1 - e^{-\gamma \tau}}{\gamma \tau} \right]$$

where

$$\Lambda = \frac{2\mu}{1 - \|\phi\|_1}, \quad \psi = \frac{1}{1 + \|\phi\|_1}, \quad \text{and} \quad \gamma = \alpha + \beta$$

- Closed Form formula for the mean Epps effect is too long to be exposed here, but can be computed!
Overview of the model in 2D

**THE MUTUALLY EXCITED PRICE MODEL (MEP) IN 2D**

\[ X(t) = N_1(t) - N_2(t) \quad \text{and} \quad Y(t) = N_3(t) - N_4(t) \]  

(9)

where \( N_i(t) \) are Hawkes processes with random intensities \( \lambda_i(t) \) given by:

\[
\lambda_{X}(t) = \frac{\mu_{X}}{2} + \int_{0}^{t} \phi_{X,X}(t-s) dN_{X}^{\pm}(s) + \int_{0}^{t} \phi_{X,Y}(t-s) dN_{Y}^{\pm}(s) 
\]  

(10)

and

\[
\lambda_{Y}(t) = \frac{\mu_{Y}}{2} + \int_{0}^{t} \phi_{Y,Y}(t-s) dN_{Y}^{\pm}(s) + \int_{0}^{t} \phi_{Y,X}(t-s) dN_{X}^{\pm}(s) 
\]  

(11)
Numerical Validation
**Figure 4:** Simulation of a Hawkes' process for $\alpha = 0.024$, $\beta = 0.11$ and $\mu = 0.016$
\[
\hat{\theta}_{reg} = \text{Argmin}_\theta | \hat{V}(\tau) - V(\tau)|^2
\]

**constraints:** \( \mu > 0, \alpha > 0, \beta > 0 \) and \( \frac{\alpha}{\beta} < 1 \)

- \( \hat{\alpha} = 0.024 \)
- \( \hat{\beta} = 0.11 \)
- \( \hat{\mu} = 0.016 \)

**Figure 5:** Signature Plot computed in link with figure 4
**Figure 6:** Simulation of a Hawkes’ process for different parameters
Figure 7: Empirical and theoretical Epps effect
Real Life Example
**Definition of Assets and Assumptions**

- **Euro-Bund futures** contracts or **Euro-Bobl futures** contracts
- Tick-by-tick (0.01) **last traded price**
- Trading time between 8 AM and 10 PM (intraday seasonality)

- **Assumption 1:** parameters of the model are **piecewise constant** in the intraday regime and the distribution of parameters is also constant for every day
  - ⇒ **stationarity** and restriction to the time period 9 AM to 11 AM from 01/11/2009 to 15/12/2009 (21 days) on the contract maturity 12/2009

- **Assumption 2:** Only jumps of size \( \pm = 1 \)
  - ⇒ Price dynamics is marginally **modified**
**Mean reverting behavior**

**Figure 8:** Histograms of jumps frequencies following a given type of jump, revealing a **mean reverting behavior**
Figure 9: Histograms of jumps following the jumps with an amplitude greater than 1.
**Figure 10:** Price of the Euro-Bobl during the whole day 11/03/2009 (maturity 12/2009)

**Figure 11:** Empirical mean signature plot computed for November’s opening day for buy orders with jumps of size ±1
**Figure 12:** Last traded price prices path of the Euro-Bobl and Euro-Bund contract on the 3rd of November 2009 (maturity 12/2009)
Figure 13: Self and cross correlations between Euro-Bobl and Euro-Bund
Conclusion and Outlooks
**Advantages**
- Simple tick-by-tick price model based on (mutually exciting) Hawkes processes
- Closed form expressions for second order properties at all time scales
- Ability to recover major high frequency stylized facts (Signature plot and Epps effect)

⇒ Simple tool to investigate intraday market features

**Downsides**
- Constant parameters $\mu$, $\alpha$ and $\beta$
- Large Number of parameters when couplings assets
- No volume

**Outlooks**
- Implementation of a high frequency trading strategy
- Backtest
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