Diversification Reconsidered: Minimum Tail Dependency

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Contents

Diversification

Overview

- 60th anniversary of MPT (see Markowitz, 1952)
- Reducing risk by investing in a variety of assets
- At least two scopes of the word ‘diversification’
  - Divers with respect to what?
  - How to measure diversification?

Diversification

Portfolio Concepts: The Peers

- Global Minimum Variance (see Markowitz, 1952, 1956, 1991): Based on Variance-Covariance
- Equal Risk Contributed (see Qian, 2005, 2006; Maillard et al., 2010; Qian, 2011): Based on variance-covariance, marginal risk contributions are equated
- CVaR Contributed (see Boudt et al., 2010, 2011): Based on downside risk measure, budgeting contributions to CVaR
- Most Diversified (see Choueifaty and Coignard, 2008; Choueifaty et al., 2011): Based on (i) correlation matrix and (ii) re-scaling of weights according to assets' riskiness
- Optimal Tail Dependent: (i) Minimum tail dependent allocation, (ii) Selection of portfolio constituents from a set of assets

Tail Dependence

Definition (i)

- Associated to Copula-concept
- Conditional probability statement for two random variables (X, Y) with marginal distributions F_X and F_Y.
- Upper tail dependence:
  $\lambda_u = \lim_{q \nearrow 1} \mathbb{P}(Y > F_Y^{-1}(q) | X > F_X^{-1}(q))$
- Lower tail dependence:
  $\lambda_l = \lim_{q \searrow 0} \mathbb{P}(Y \leq F_Y^{-1}(q) | X \leq F_X^{-1}(q))$

Tail Dependence

Definition (ii)

- Expressed in Copula-terms:
- Upper tail dependence:
  $\lambda_u = 2 + \lim_{q \searrow 0} \frac{\mathcal{C}(1-q,1-q)-1}{q}$
- Lower tail dependence:
  $\lambda_l = \lim_{q \searrow 0} \frac{\mathcal{C}(q,q)}{q}$
- Student's t Copula:
  $\lambda_u = \lambda_l = 2t_{\nu+1}(-\sqrt{\nu+1}\sqrt{(1-\rho)/(1+\rho)})$
- Archimedean Copulae:
  - Gumbel Copula: $\lambda_u = 2 - 2^{1/\theta}$
  - Clayton Copula: $\lambda_l = 2^{-1/\delta}$

Tail Dependence

Non-Parametric Estimators (i)

- Synopsis of estimators in Dobrić and Schmid (2005); Frahm et al. (2005); Schmidt and Stadtmüller (2006)
- Focus on lower tail dependence (losses for long-only)
- Based on empirical copula of N pairs $(X_1, Y_1), \ldots, (X_N, Y_N)$ with corresponding order statistics $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(N)}$ and $Y_{(1)} \leq Y_{(2)} \leq \ldots \leq Y_{(N)}$
- Empirical Copula:
  $\mathcal{C}_N(\frac{i}{N}, \frac{j}{N}) = \frac{1}{N} \sum_{l=1}^{N} I(X_l \leq X_{(i)} \wedge Y_l \leq Y_{(j)})$
  with $i, j = 1, \ldots, N$ and I is the indicator function, which takes a value of one, if the condition stated in parenthesis is true.

Tail Dependence

Non-Parametric Estimators (ii)

- Estimators depend on threshold parameter k
- Estimators are consistent and unbiased, if $k \sim \sqrt{N}$ (see Dobrić and Schmid, 2005)

1. Secant-based: $\lambda_L^{(1)}(N, k) = \left[\frac{k}{N}\right]^{-1} \cdot \mathcal{C}_N\left(\frac{k}{N}, \frac{k}{N}\right)$
2. Slope-based: $\lambda_L^{(2)}(N, k) = \left[\sum_{i=1}^{k}\left(\frac{i}{N}\right)^2\right]^{-1} \cdot \sum_{i=1}^{k}\left[\frac{i}{N} \cdot \mathcal{C}_N\left(\frac{i}{N}, \frac{i}{N}\right)\right]$
3. Mixture-based: $\lambda_L^{(3)}(N, k) = \frac{\sum_{i=1}^{k}\left(\mathcal{C}_N\left(\frac{i}{N}, \frac{i}{N}\right) - \left(\frac{i}{N}\right)^2\right)\left(\left(\frac{i}{N}\right) - \left(\frac{i}{N}\right)^2\right)}{\sum_{i=1}^{k}\left(\frac{i}{N} - \left(\frac{i}{N}\right)^2\right)^2}$

Tail Dependence

Utilization in Optimization

- Minimum Tail Dependent Portfolio
  - Approach similar to MDP
  - First step: Derive optimal solution if TDC-matrix is used with main-diagonal elements are set to one.
  - Second step: Re-scale optimal weight vectors by assets volatility (riskiness).
  - Implemented in package `FRAPO` (see Pfaff, 2012)
- Asset Selection
  - Benchmark-relative Optimisations
  - Choose constitutents which are least lower tail dependent to the benchmark (index).
  - No implication with respect to the upper tail dependencies, in contrast to low β strategies that are in general based on a symmetric co-dispersion measure.

MTD vs. Peer-Strategies

Overview

- Swiss Performance Sector Indexes
- Static long-only optimisation according to
  - GMV
  - MDP
  - ERC
  - MTD
- Analysis of allocations, risk- & marginal risk contributions, and key measures

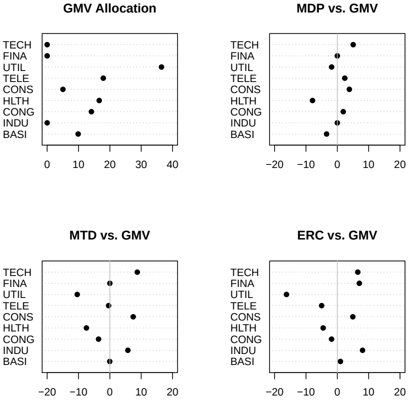
MTD vs. Peer-Strategies

Optimisations

```
> library(FRAPO)
> library(fPortfolio)
> library(lattice)
> ## Loading data and calculating returns
> data(SPISECTOR)
> Idx <- interpNA(SPISECTOR[, -1], method = "before")
> R <- returnseries(Idx, method = "discrete", trim = TRUE)
> V <- cov(R)
> ## Portfolio Optimisations
> GMVw <- Weights(PGMV(R))
> MDPw <- Weights(PMD(R))
> MTDw <- Weights(PMTD(R))
> ERCw <- Weights(PERC(V))
> ## Graphical displays of allocations
> oldpar <- par(no.readonly = TRUE)
> par(mfrow = c(2, 2))
> dotchart(GMVw, xlim = c(0, 40), main = "GMV Allocation", pch = 19)
> dotchart(MDPw - GMVw, xlim = c(-20, 20), main = "MDP vs. GMV", pch = 19)
> abline(v = 0, col = "gray")
> dotchart(MTDw - GMVw, xlim = c(-20, 20), main = "MTD vs. GMV", pch = 19)
> abline(v = 0, col = "gray")
> dotchart(ERCw - GMVw, xlim = c(-20, 20), main = "ERC vs. GMV", pch = 19)
> abline(v = 0, col = "gray")
> par(oldpar)
```

MTD vs. Peer-Strategies

Graphical displays of allocations



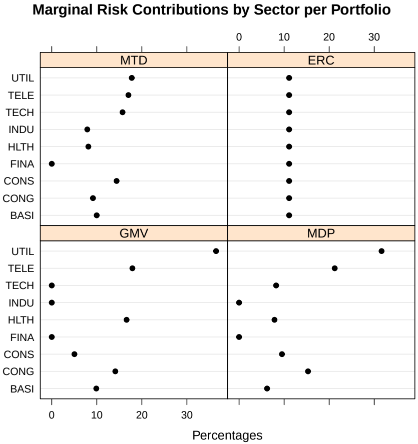
MTD vs. Peer-Strategies

Marginal Risk Contributions

```
> ## Combining solutions
> W <- cbind(GMVw, MDPw, MTDw, ERCw)
> ## MRC
> MRC <- apply(W, 2, mrc, Sigma = V)
> rownames(MRC) <- colnames(Idx)
> colnames(MRC) <- c("GMV", "MDP", "MTD", "ERC")
> ## lattice plots of MRC
> Sector <- factor(rep(rownames(MRC), 4), levels = sort(rownames(MRC)))
> Port <- factor(rep(colnames(MRC), each = 9), levels = colnames(MRC))
> MRCdf <- data.frame(MRC = c(MRC), Port, Sector)
> dotplot(Sector ~ MRC | Port, groups = Port, data = MRCdf,
+         xlab = "Percentages",
+         main = "Marginal Risk Contributions by Sector per Portfolio",
+         col = "black", pch = 19)
> dotplot(Port ~ MRC | Sector, groups = Sector, data = MRCdf,
+         xlab = "Percentages",
+         main = "Marginal Risk Contributions by Portfolio per Sector",
+         col = "black", pch = 19)
```

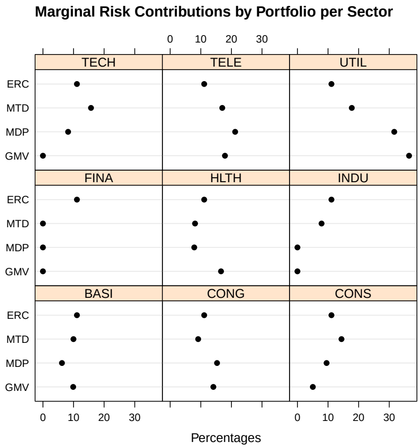
MTD vs. Peer-Strategies

Graphical displays of MRC (i)



MTD vs. Peer-Strategies

Graphical displays of MRC (ii)



MTD vs. Peer-Strategies

Portfolio Characteristics

Measures	GMV	MDP	MTD	ERC
Standard Deviation	0.813	0.841	0.903	0.949
ES (modified, 95 %)	2.239	2.189	2.313	2.411
Diversification Ratio	1.573	1.593	1.549	1.491
Concentration Ratio	0.218	0.194	0.146	0.117

Table: Key measures of portfolio solutions for SPI sectors

Low Tail Dependency vs. Low Beta

Overview

- Benchmark relative optimisation: S&P 500
- Weekly data: 291 observations of the index and 457 constituents. The sample starts in March 1991 and ends in September 1997. Source: `INDTRACK6` (OR-Library)
- Long-only portfolio, in-sample period 260 observations
- Similar analysis in Malevergne and Sornette (2008)

Low Tail Dependency vs. Low Beta

Backtest I: Data Preparation

```
> library(FRAPO)
> library(copula)
> ## S&P 500
> data(INDTRACK6)
> ## Market and Asset Returns
> RM <- returnseries(INDTRACK6[1:260, 1], method = "discrete", trim = TRUE)
> RA <- returnseries(INDTRACK6[1:260, -1], method = "discrete", trim = TRUE)
> ## Beta of S&P 500 stocks
> Beta <- apply(RA, 2, function(x) cov(x, RM) / var(RM))
> ## Computing Kendall's tau
> Tau <- apply(RA, 2, function(x) cor(x, RM, method = "kendall"))
> ## Clayton Copula: Lower Tail Dependence
> ThetaC <- copClayton@tauInv(Tau)
> LambdaL <- copClayton@lambdaL(ThetaC)
> ## Selecting Stocks below median; inverse log-weighted and scaled
> IdxBeta <- Beta < median(Beta)
> WBeta <- -1 * log(abs(Beta[IdxBeta]))
> WBeta <- WBeta / sum(WBeta) * 100
> ## TD
> IdxTD <- LambdaL < median(LambdaL)
> WTD <- -1 * log(LambdaL[IdxTD])
> WTD <- WTD / sum(WTD) * 100
> Intersection <- sum(names(WTD) %in% names(WBeta)) / length(WBeta) * 100
```

Low Tail Dependency vs. Low Beta

Backtest II: Out-of-sample

```
> ## Out-of-Sample Performance
> RMo <- returnseries(INDTRACK6[260:290, 1], method = "discrete",
+                     percentage = FALSE) + 1
> RAo <- returnseries(INDTRACK6[260:290, -1], method = "discrete",
+                     percentage = FALSE) + 1
> ## Benchmark
> RMo[1] <- 100
> RMEquity <- cumprod(RMo)
> ## Low Beta
> LBEquity <- RAo[, IdxBeta]
> LBEquity[1, ] <- WBeta
> LBEquity <- rowSums(apply(LBEquity, 2, cumprod))
> ## TD
> TDEquity <- RAo[, IdxTD]
> TDEquity[1, ] <- WTD
> TDEquity <- rowSums(apply(TDEquity, 2, cumprod))
```

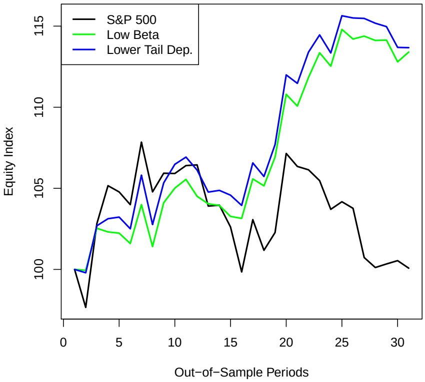
Low Tail Dependency vs. Low Beta

Backtest III: Progression of Portfolio Equity

```
> ## Collecting results
> y <- cbind(RMEquity, LBEquity, TDEquity)
> ## Time series plots of equity curves
> plot(RMEquity, type = "l", ylim = range(y), ylab = "Equity Index",
+      xlab = "Out-of-Sample Periods")
> lines(LBEquity, col = "green")
> lines(TDEquity, col = "blue")
> legend("topleft", legend = c("S&P 500", "Low Beta", "Lower Tail Dep."),
+        col = c("black", "green ", "blue"))
> ## Bar plot of out-performance
> RelOut <- rbind((LBEquity / RMEquity - 1) * 100,
+                 (TDEquity / RMEquity - 1) * 100)
> RelOut <- RelOut[, -1]
> barplot(RelOut, beside = TRUE, ylim = c(-5, 17), names.arg = 1:ncol(RelOut),
+         legend.text = c("Low Beta", "Lower Tail Dep."),
+         args.legend = list(x = "topleft"))
> abline(h = 0)
> box()
```

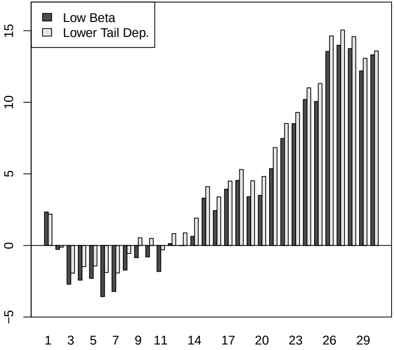
Low Tail Dependency vs. Low Beta

Backtest IV: Graphical Displays



Low Tail Dependency vs. Low Beta

Backtest IV: Graphical Displays



Outlook

Extension and Modifications

- Use lower-partial moments for re-scaling of weights
- Use upper- /lower TD ratio for optimization
- Adapt approach to long-/short strategies

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