COMPUTATIONAL FINANCE & RISK MANAGEMENT
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ROBUST STATISTICS IN PORTFOLIO CONSTRUCTION

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1. Why robust statistics in finance?
2. R robust library
3. Robustness concepts
4. Robust factor models
5. Robust covariance and correlation
6. Robust volatility clustering models

Appendix: R robust library inference

References

- Maronna, R. Martin, R.D., and Yohai, V. J. (2006). *Robust Statistics* : *Theory and Methods*, Wiley.
- Martin, R. D., Clark, A and Green, C. G. (2010). "Robust Portfolio Construction", in *Handbook of Portfolio Construction: Contemporary Applications of Markowitz Techniques*, J. B. Guerard, Jr., ed., Springer.
- Bailer, H., Maravina, T. and Martin, R. D. (2012). "Robust Betas for Asset Managers", in *The Oxford Handbook of Quantitative Asset Management*, Scherer, B. and Winston, K., editors, Oxford University Press.
- Chapter 6 of Scherer, B. and Martin, R. D. (2004). Modern Portfolio Construction, Springer

"For years I have utilized your robust analysis … to prepare and trade portfolios. My clients depend upon these robust results, so thank you for all your pioneering work in the field."

*Paul Lasky, P & B Consultants**

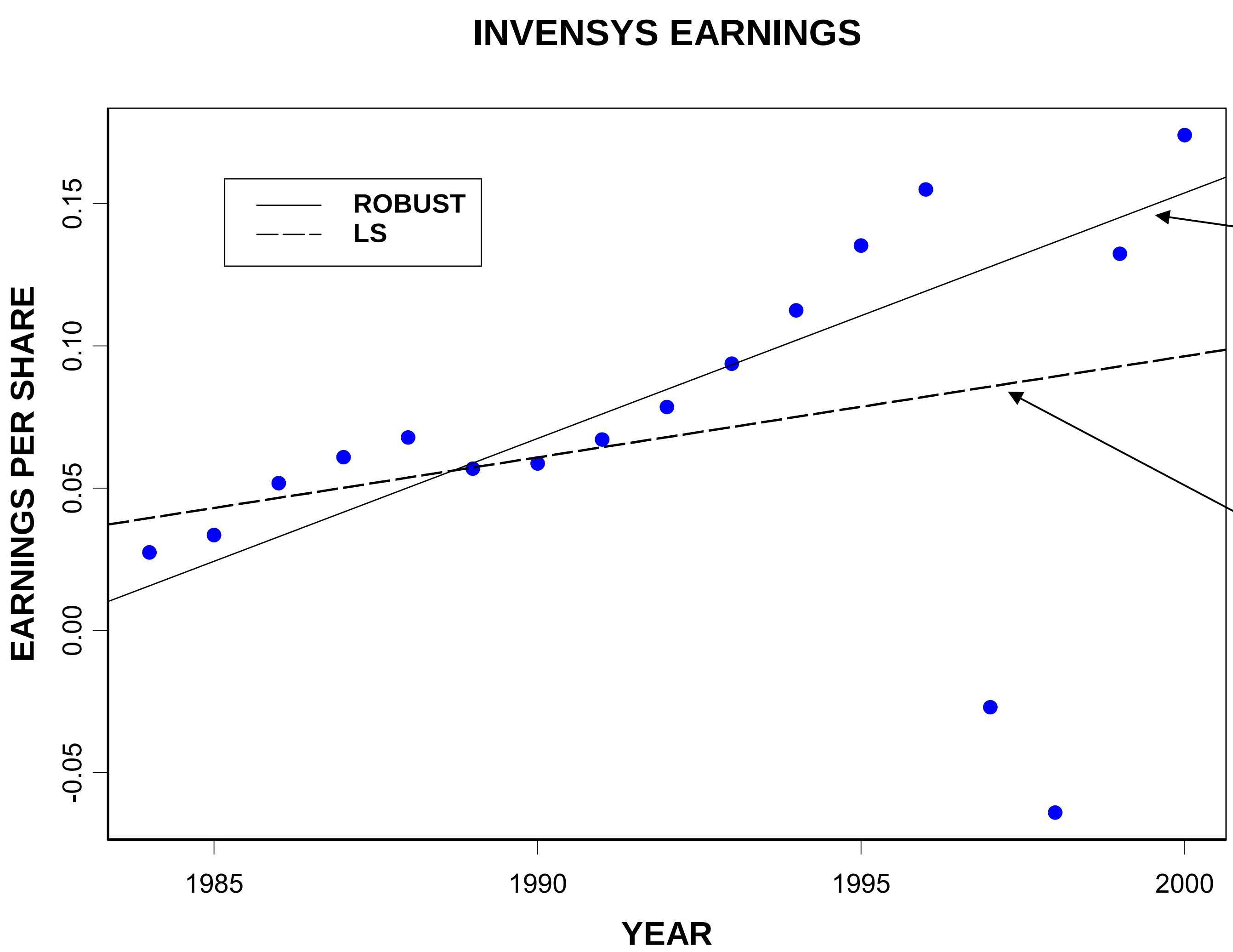
* Unsolicited email from Paul Lasky to Doug Martin, Oct. 2004

1. Why Robust Statistics in Finance?

All classical estimates are vulnerable to extreme distortion by outliers, and financial data has outliers to various degrees. You need robust estimates that are:

- Not much influenced by outliers
- Good model fit to bulk of the data
- Reliable returns outlier detection
- Stable inference and prediction

Robust vs. LS Prediction of EPS

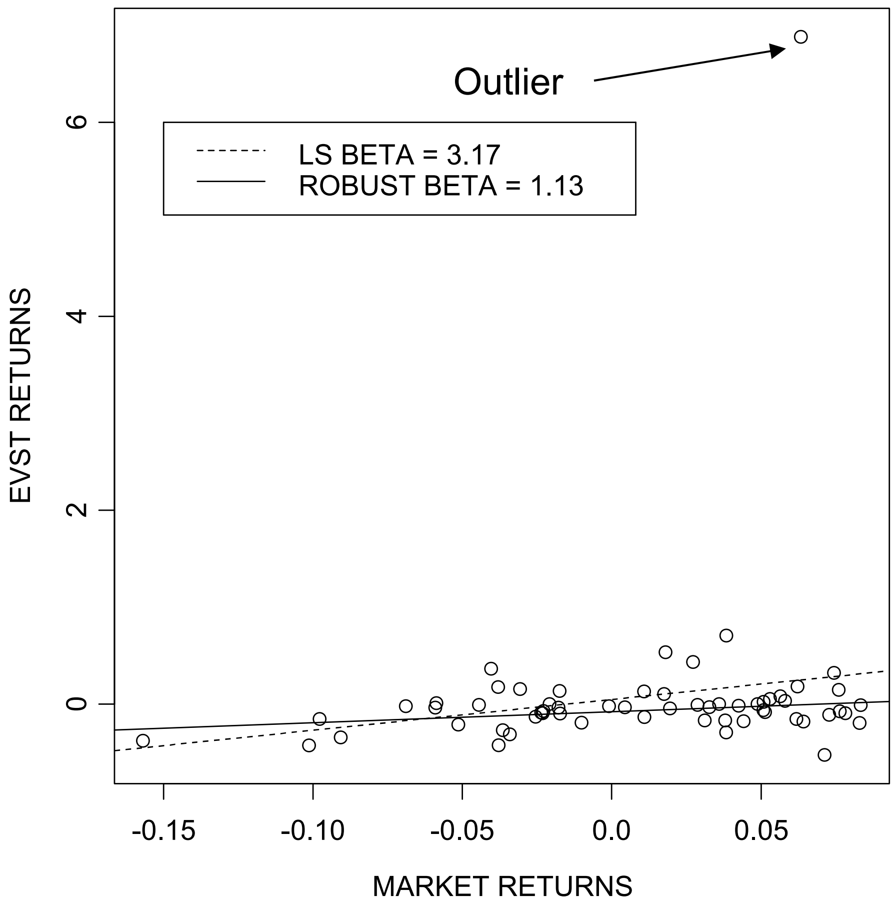


The ROBUST line fits bulk of data well by down-weighting the 2-D outliers, and provides a good predictor of EPS

The CLASSIC LEAST SQUARES line is a poor fit to the bulk of the data, and is a very poor predictor of EPS

Example from S-PLUS user in DuPont Corporate Finance

Robust vs LS Betas



$$r_t = \alpha + r_{m,t}\beta + \varepsilon_t, \quad t = 1, \cdots, T$$

CLASSICAL BETA = 3.17

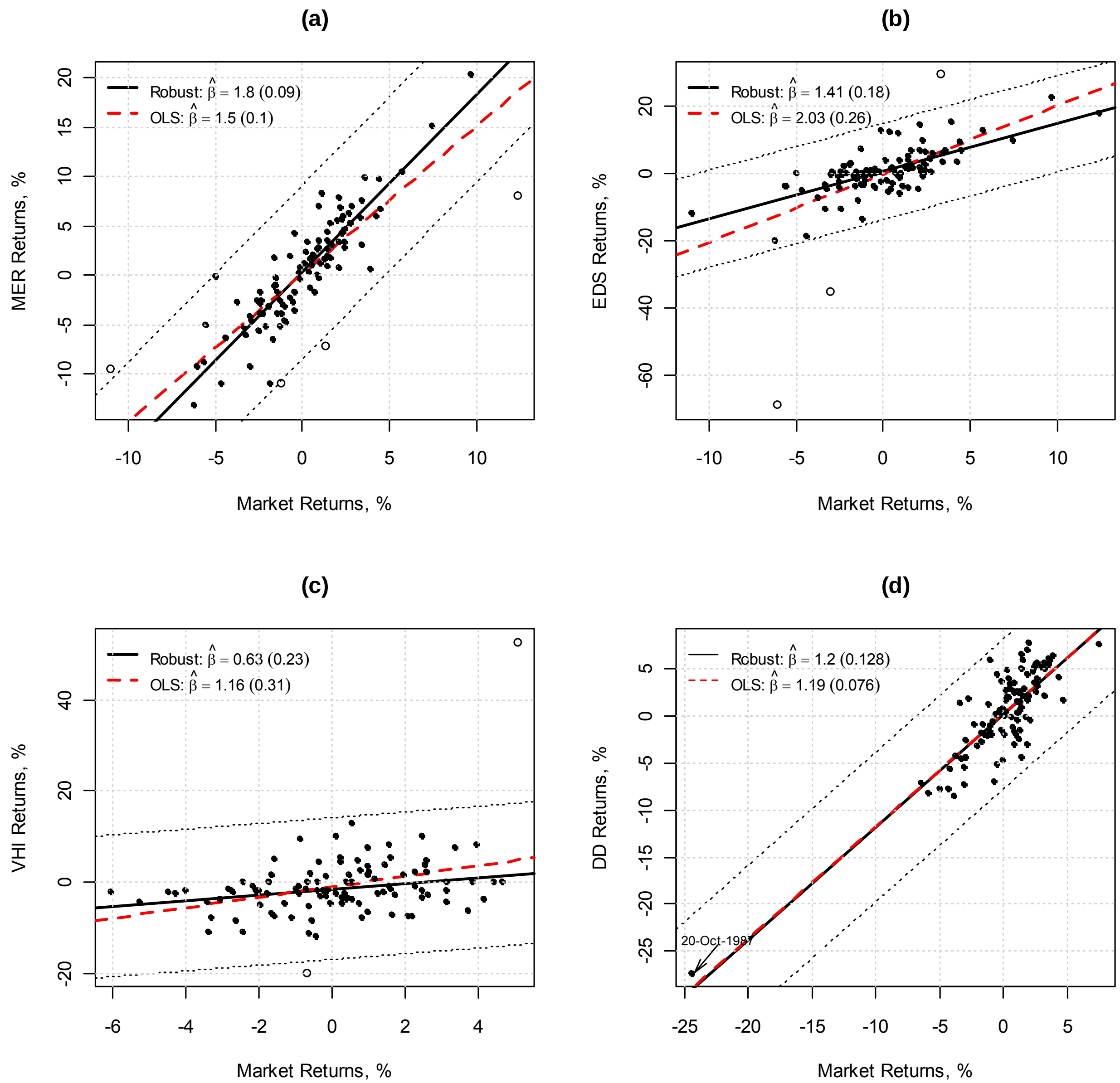
This is what you get today: a very misleading result, caused by one outlier, and predicts future betas poorly!

ROBUST BETA = 1.13

Fits bulk of data to accurately describe the typical risk and return, and predicts future betas.

See Martin and Simin (2003), Financial Analysts Journal

Robust vs. LS Market Models*

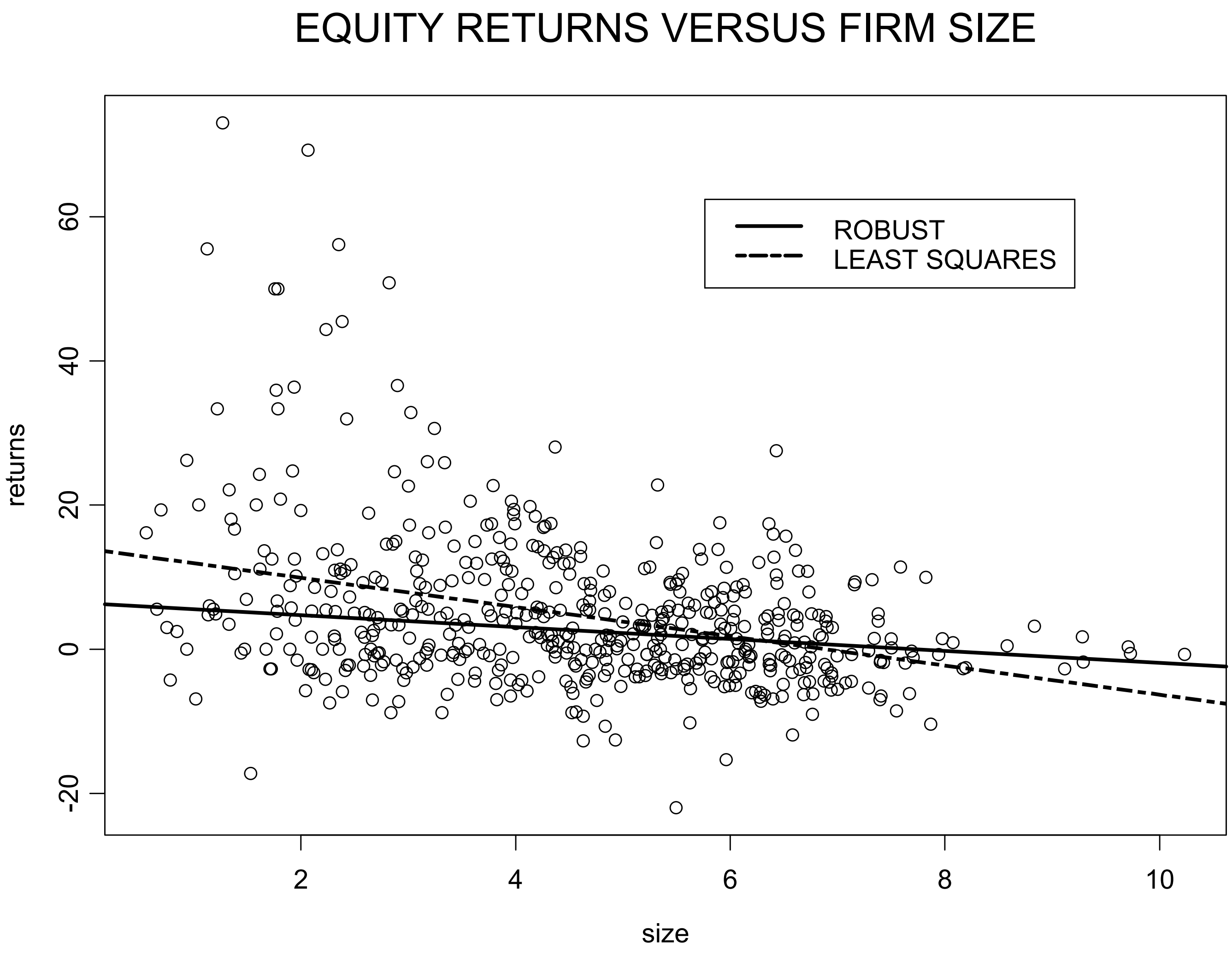


* Bailer, Maravina and Martin (2012)

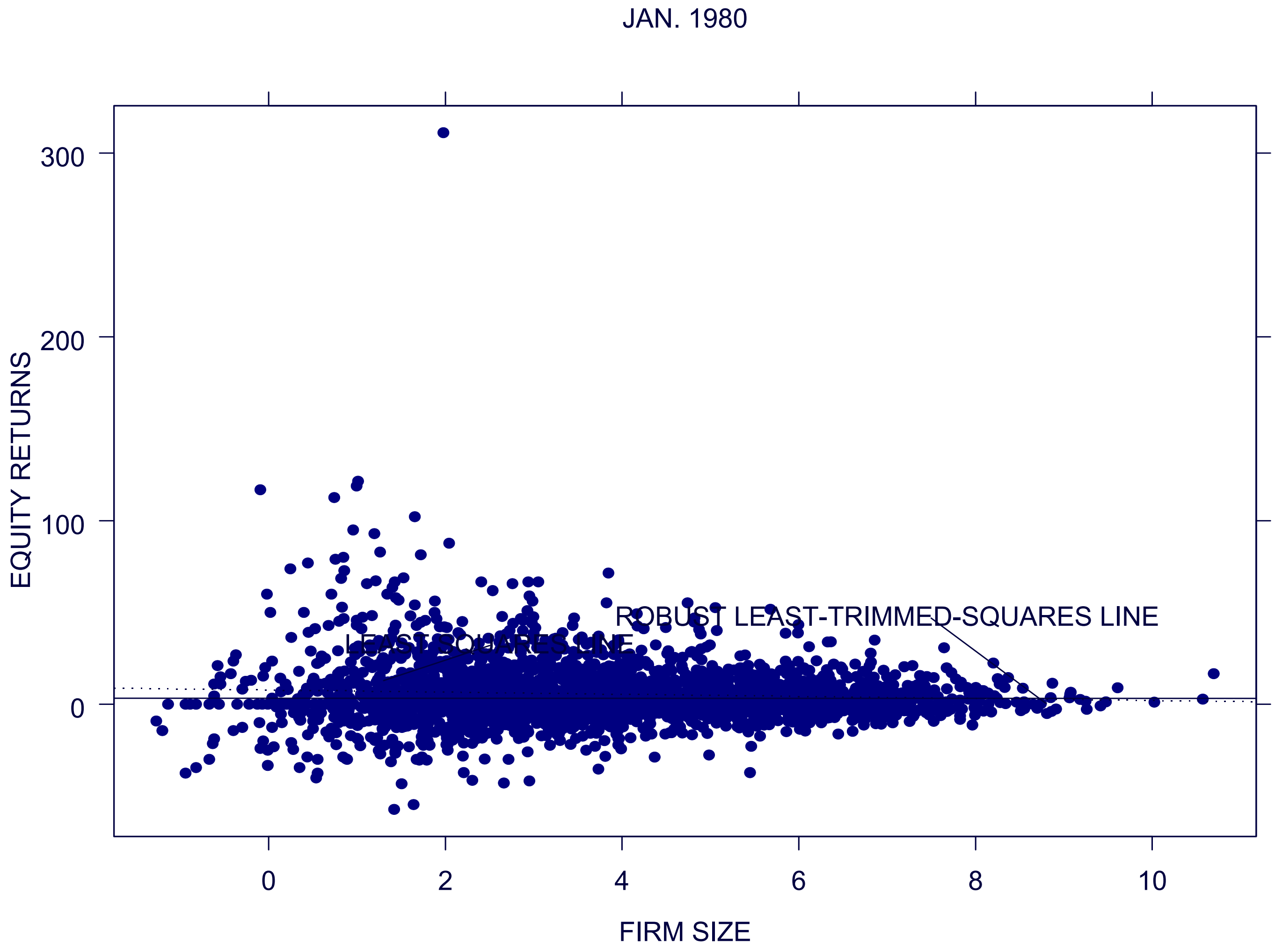
Robust Regression of Returns vs. Size

- Fama and French (1992) results: equity returns are **negatively** related to firm size when using LS
- Knez and Ready (1997) results: returns are **positively** related to size for the vast majority of the data when using LTS regression
  - Positive relationship is rather constant for all trimming between 50% and 5%, and the relationship remains positive even at 1% trimming. Furthermore there is little loss of efficiency in the 1% to 5% trimming range

Monthly Returns July 1963

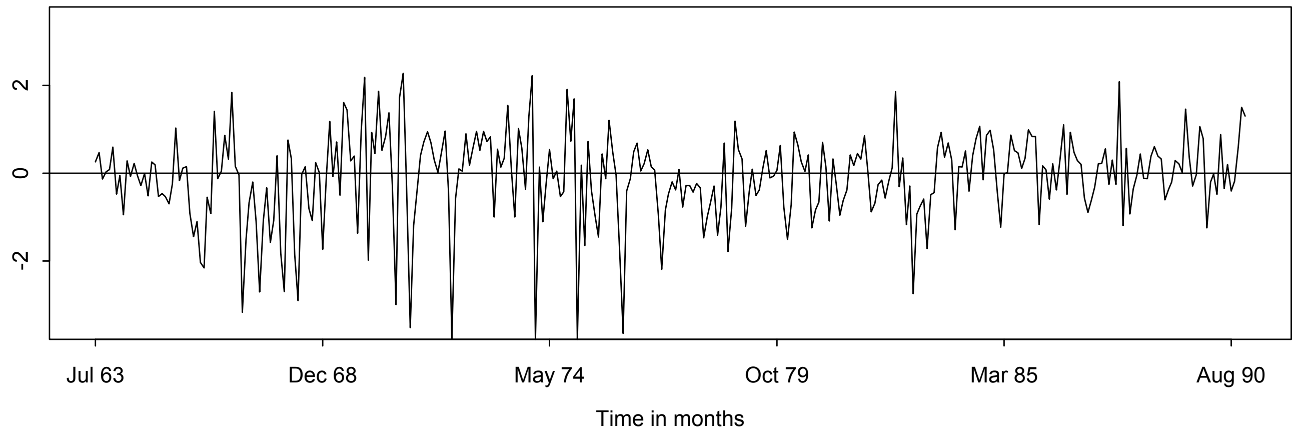


Monthly Returns January 1980



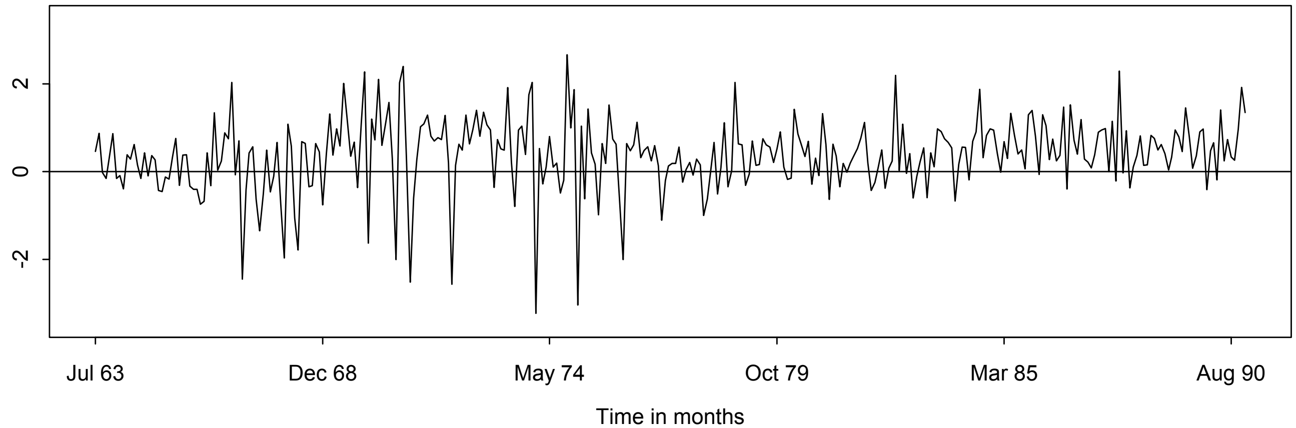
Time Series of Slope Coefficients

LS SLOPE COEFFICIENTS: Regressions of Returns on Size



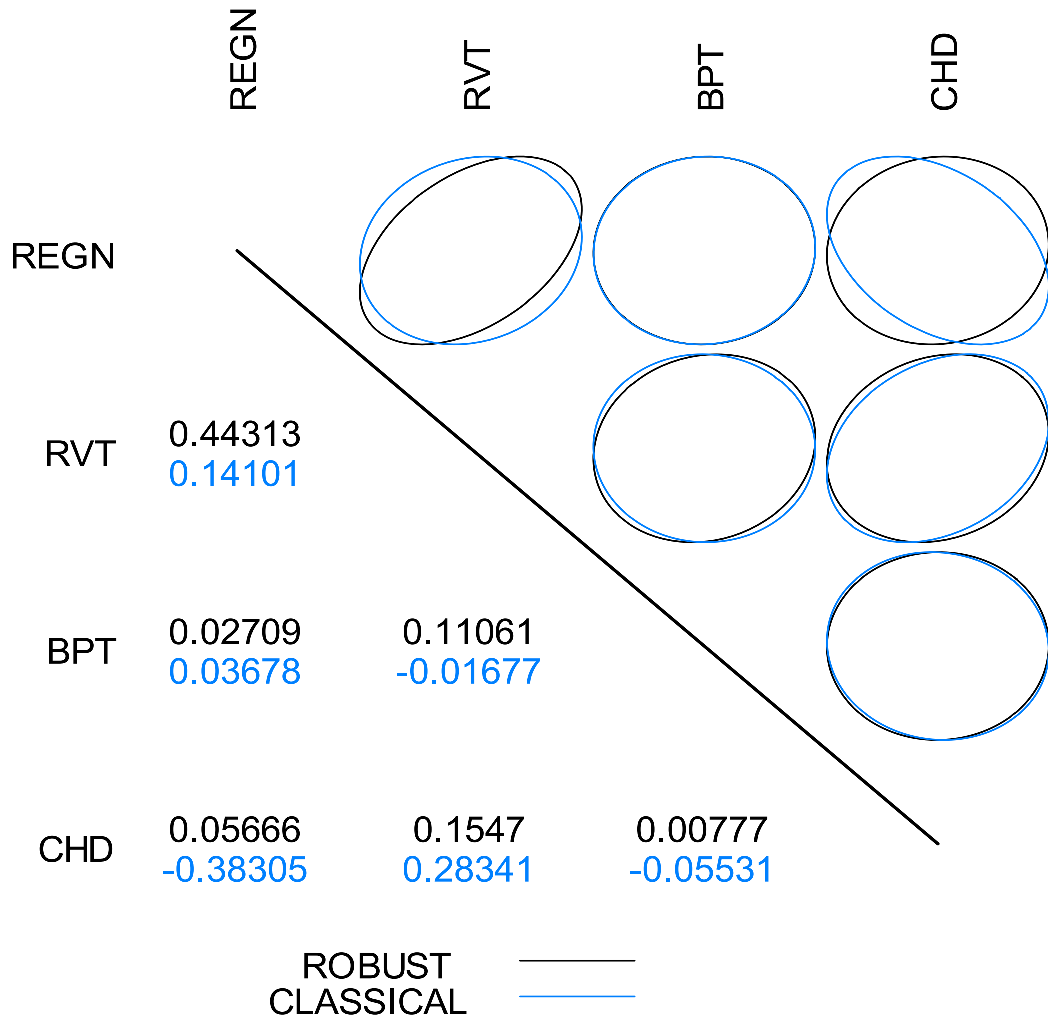
Fama-French:
t-statistic indicates
negative relation

ROBUST LTS SLOPE COEFFICIENTS: Regressions of Returns on Size



Knez-Ready:
t-statistic indicates
positive relation
(for vast majority
of firms)!

Robust Correlations for Small-Caps



Important Points

- **Myth**: Robust statistical methods throw away what may be the most important data values
  - Model fit is only the first step in the analysis, and robust statistics reliably detects outliers that are often completely missed with classical estimates. Then can check if they are important or just noise.
- **Do not use blindly in risk applications**
  - Outlier rejection can give misleading optimism about risk. But they can sometimes lead to pessimism.

2. R "robust" Package

John Tukey (1979): "… just which robust methods you use is not important – what is important is that you use some. It is perfectly proper to use both classical and robust methods routinely, and only worry when they differ enough to matter. But when they differ, you should think hard."

Robust Library Motivation: Make it possible for everyone to easily compute classical and robust estimates!

Cavet : It is important which robust method you use! Some are much better than others. Need some theory.

R Package “robust”

Originally created by Insightful under an NIH SBIR grant, with Doug Martin as P.I., Kjell Konis (primary) and Jeff Wang as developers. Given over to R by Insightful with Kjell Konis as maintainer. The CRAN site shows:

Author:	Jiahui Wang, Ruben Zamar, Alfio Marazzi, Victor Yohai, Matias Salibian-Barrera, Ricardo Maronna, Eric Zivot, David Rocke, Doug Martin, Martin Maechler, Kjell Konis.
Maintainer:	Kjell Konis <kjell.konis at me.com>

There is also the R package “robustbase”, with a lot of people working on it. Kjell has a goal of putting as many of the “robust” library methods (current and future) into “robustbase” as he can manage. I

Original Goals

- Automatic computation of both classical and robust estimates
- Reliable multivariate outlier detection
- Trellis graphics comparisons plots
- Robust as well as classical statistical inference
- Scalable methods for robust regression and covariance

Modeling Methods

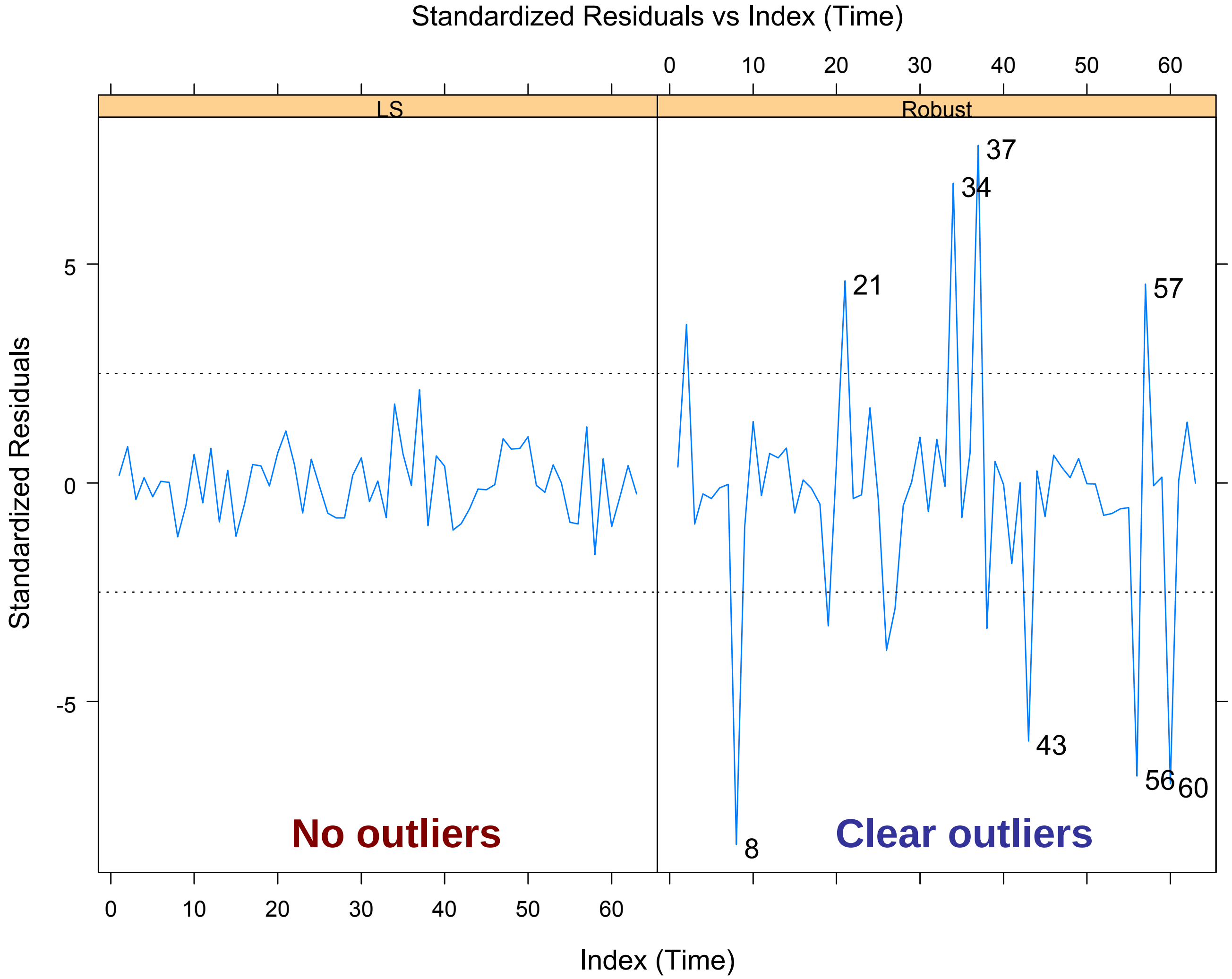
- Robust Linear Regression and Model Selection
- Robust ANOVA
- Robust Covariance and Correlation Estimation
- Robust Principal Component Analysis
- Robust Fitting of Poisson and Logistic GLIM's
- Robust Discriminant Analysis
- Robust Parameter Estimates for Asymmetric Distributions

Example: Robust vs. LS Regression

The Wagner Data (Hubert and Rousseeuw, 1997)

Y :	rate of unemployment
PA:	percentage engaged in production activities (PA)
GPA:	growth in PA
HS:	percentage engaged in higher services (HS)
GHS:	growth in HS
Region:	geographical region around Hannover (21 regions)
Period:	time period (3 periods: ’79-’82, ’83-’88, ’89-’92)

Least Squares Gives No Clue!

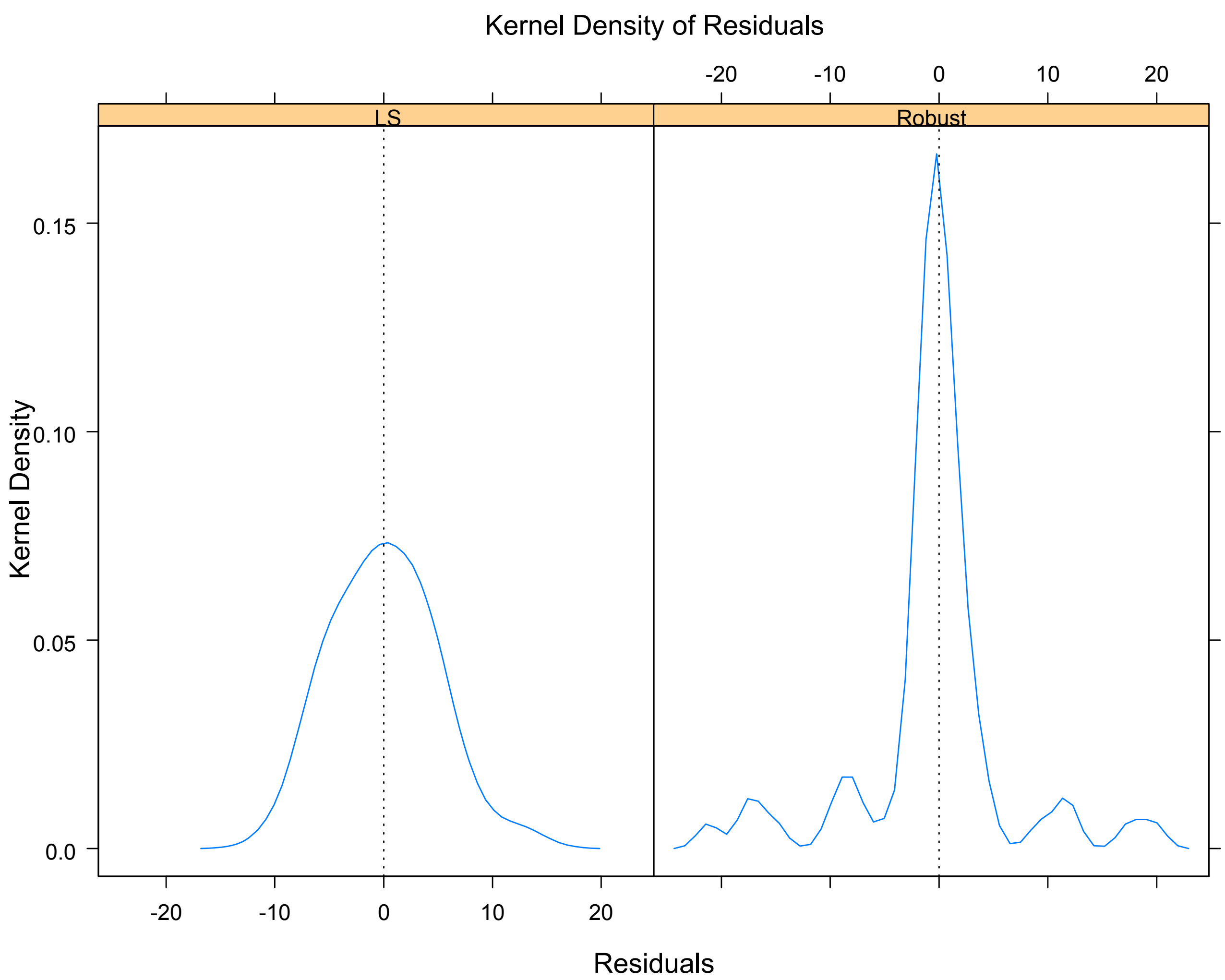


LS Fit

Approximately normal error distribution

Robust Fit

Non-normal errors, more compact central scale

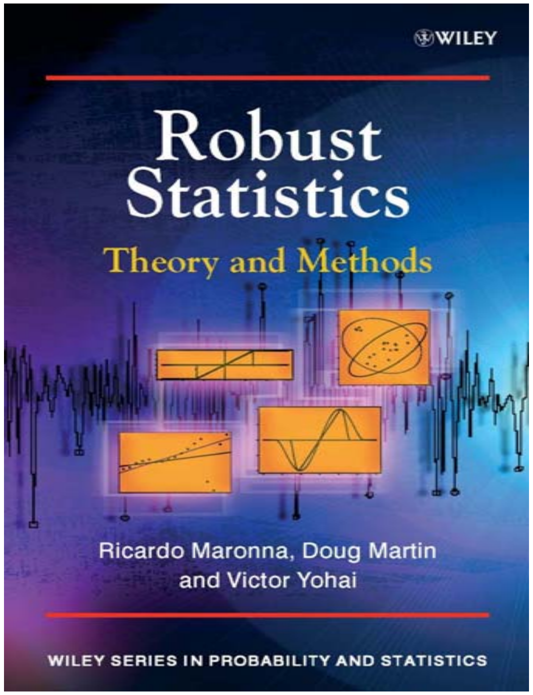


#Compare robust fit with least squares fit

```
stack.fm <- fit.models(list(Robust = "lmRob",
           "Least Squares“ = "lm"),
            Loss ~ ., data = stack.dat)
stack.fm
summary(stack.fm)
plot(stack.fm) #Choice 1: All
```

3. ROBUSTNESS THEORY

- **Efficiency Robustness**
- **Bias Robustness**
- **Other Concepts**
  - **Min-max robustness**
  - **Continuity**
  - **Bounded influence**
  - **Breakdown point**



Standard Outlier Generating Model

Nominal parametric distribution

Unknown asymmetric, or non-elliptical distributions

$$F = (1-\gamma) \cdot F_{\boldsymbol{\theta}} + \gamma \cdot H$$

Unknown, often “smallish” (.01 to .02-.05) but want need protection for “large” values up to .5.

Robustness: Doing well near a parametric model.
In most applications $F_{\boldsymbol{\theta}}$ is a normal distribution.

Efficiency Robustness*

High efficiency when the data distribution F is normal **and also** when F is "nearly normal" with fat tails, where

$$EFF(\hat{\theta}_{ROBUST}, F) = \frac{\operatorname{var}(\hat{\theta}_{MLE}, F)}{\operatorname{var}(\hat{\theta}_{ROBUST}, F)}$$

NOTE: Tukey favored an empirical "tri-efficiency" metric with one distribution normal, one mildly fat-tailed (normal mixture) and one very fat-tailed (Cauchy tails), and sometimes use "best known" estimate in place of MLE.

* Tukey (1960). "A Survey of Sampling from Contaminated Distributions"

Bias Robustness

Want close to smallest attainable MSE

$$\mathrm{MSE}(\gamma, T_n) = \mathrm{VAR}(\gamma, T_n) + \mathrm{B}^2(\gamma, T_n)$$

Since

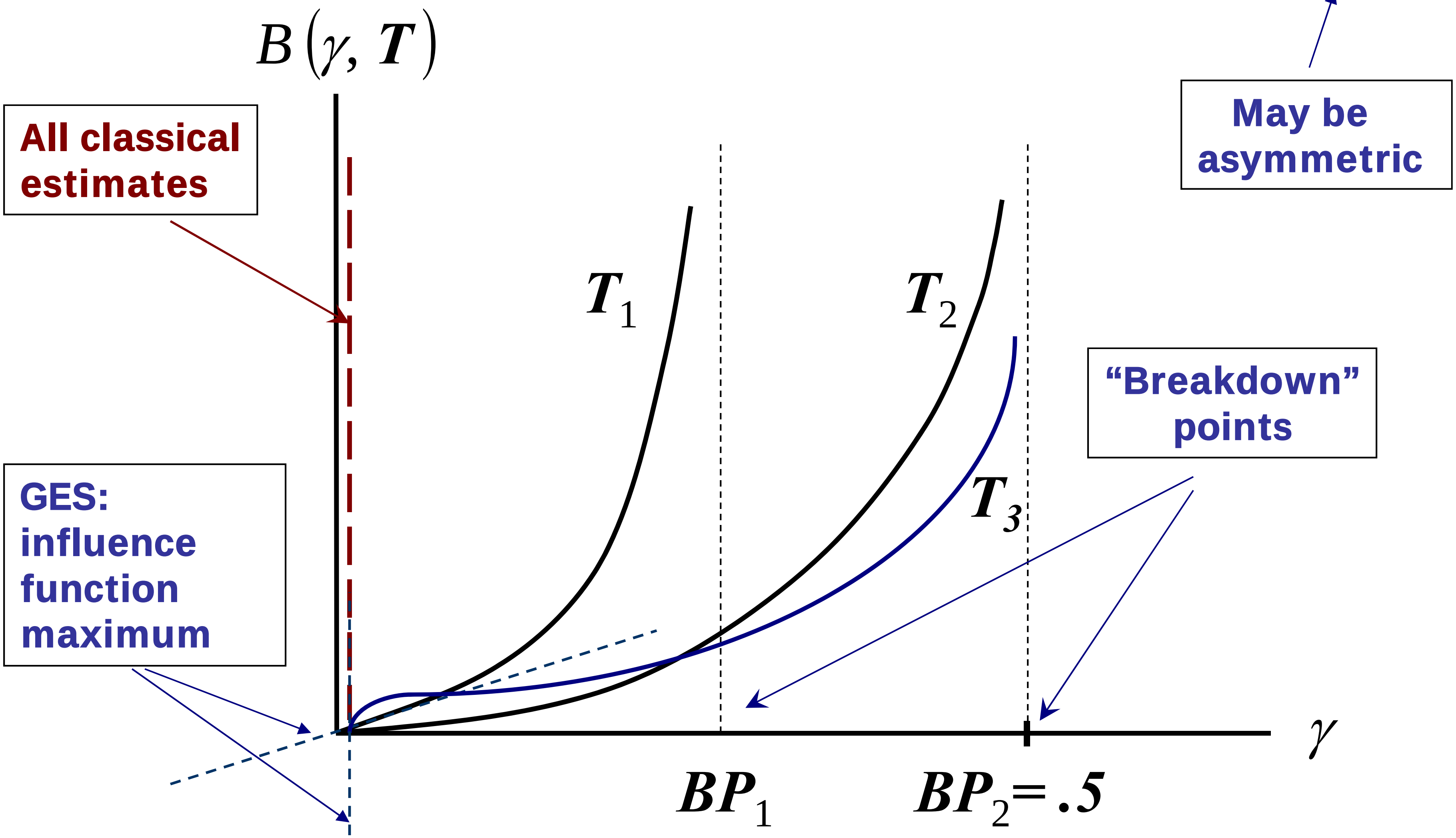
$$\mathrm{VAR}(\gamma, T_n) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty$$

Choose T_n to:

minimize $\mathrm{B}(\gamma, T_n; F)$ for $F = (1-\gamma) \cdot F_{\boldsymbol{\theta}} + \gamma \cdot H$

Maximum Bias Curves

Maximum bias of $\boldsymbol{T}$ over **all** H in $F=(1-\gamma)\cdot F_{\boldsymbol{\theta}}+\gamma\cdot H$



Min-Max Bias Robust Estimates

- **Location estimation** (Huber, 1964)
  - Sample median ("Leads to uneventful theory")
- **Scale estimation** (Martin and Zamar, 1989, 1991)
  - For nominal exponential model: adjusted median
  - Median absolute deviation about the median (MADM)*
  - Shortest half of the data (SHORTH)*
- **Regression estimation**: Next section

* Approximately for all fractions γ of outliers, exact as $\gamma \to .5$

4. ROBUST FACTOR MODELS

- More accurate decomposition of risk into factor risk and specific risk
- More accurate covariance matrix for MV portfolio optimization
- Create accurate fat-tailed skewed multivariate distribution simulation model for risk analysis and portfolio optimization with downside risk measures

Bias Robust M-Estimates

Time Series Factor Models for FoF's (factor returns known)

$$\hat{\boldsymbol{\beta}}_k = \mathrm{argmin}_{\boldsymbol{\beta}_k} \sum_{t=1}^{T} \rho\left(\frac{r_{k,t} - \mathbf{f}_t{}'\boldsymbol{\beta}_k}{\hat{s}_o}\right), \quad k = 1, \cdots, K$$

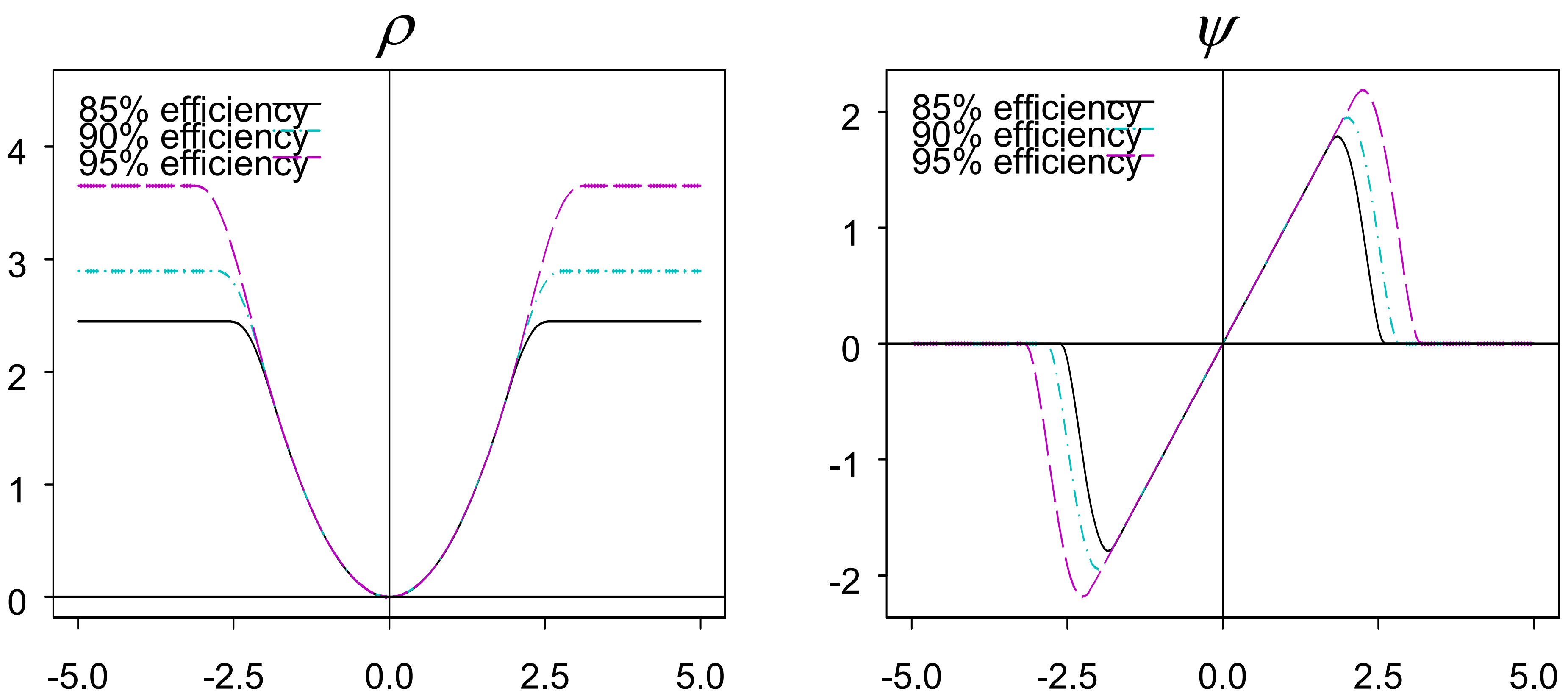
Fundamental Factor Models (exposures/betas known)

$$\hat{\mathbf{f}}_t = \mathrm{argmin}_{\mathbf{f}_t} \sum_{k=1}^{K} \rho\left(\frac{r_{k,t} - \boldsymbol{\beta}_k'\mathbf{f}_t}{\hat{s}_o}\right), \quad \mathrm{t} = 1, \cdots, K$$

ρ must be **bounded** hence **non-convex** for robustness, which requires sophisticated optimization.

Optimal* Rho and Psi

*Minimizes maximum bias due to outliers with only somewhat higher variance OLS when returns are normally distributed. See Yohai and Zamar (1997), Svarc, Yohai and Zamar (2002).

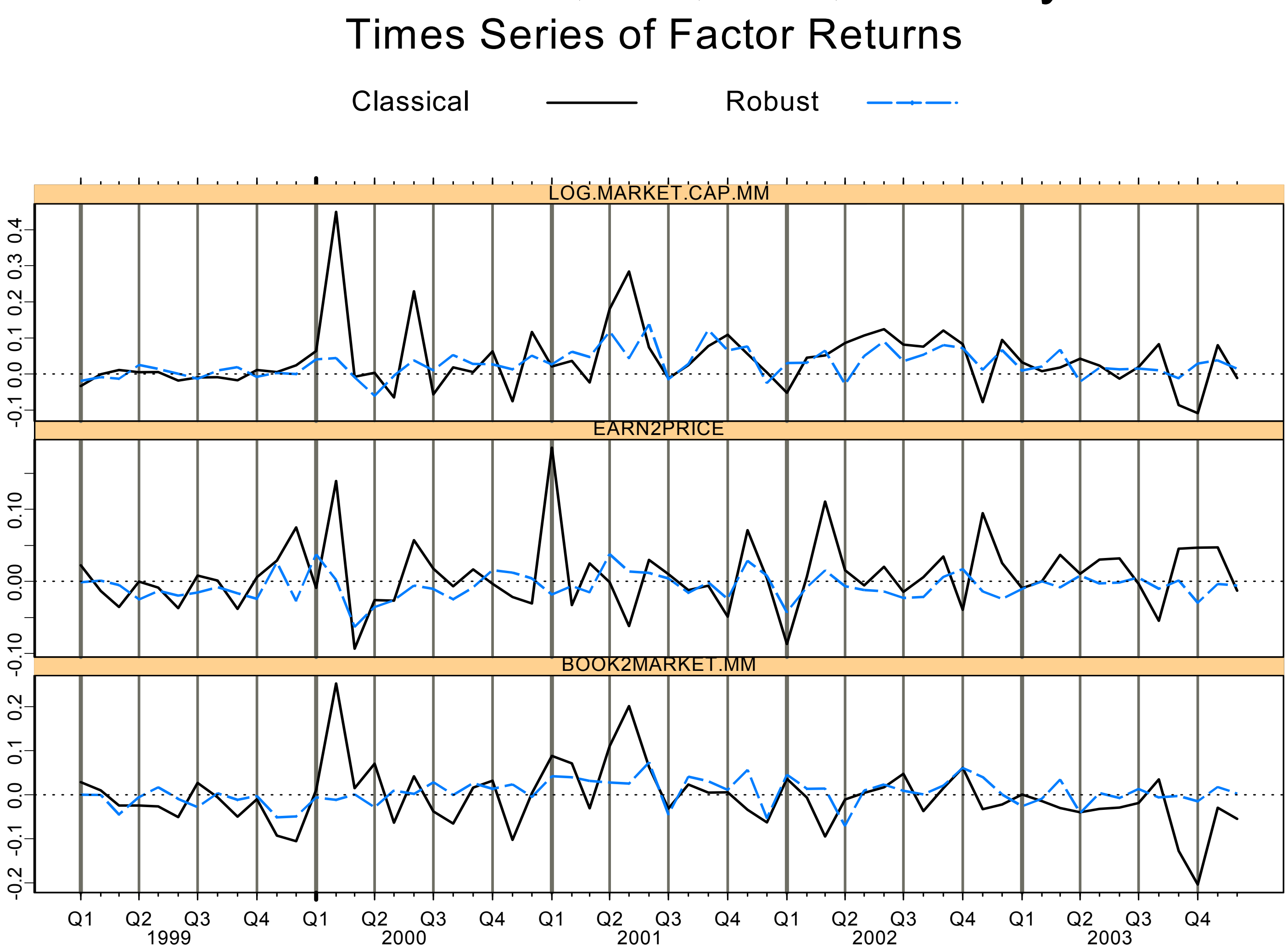


Robust Fundamental Factor Models

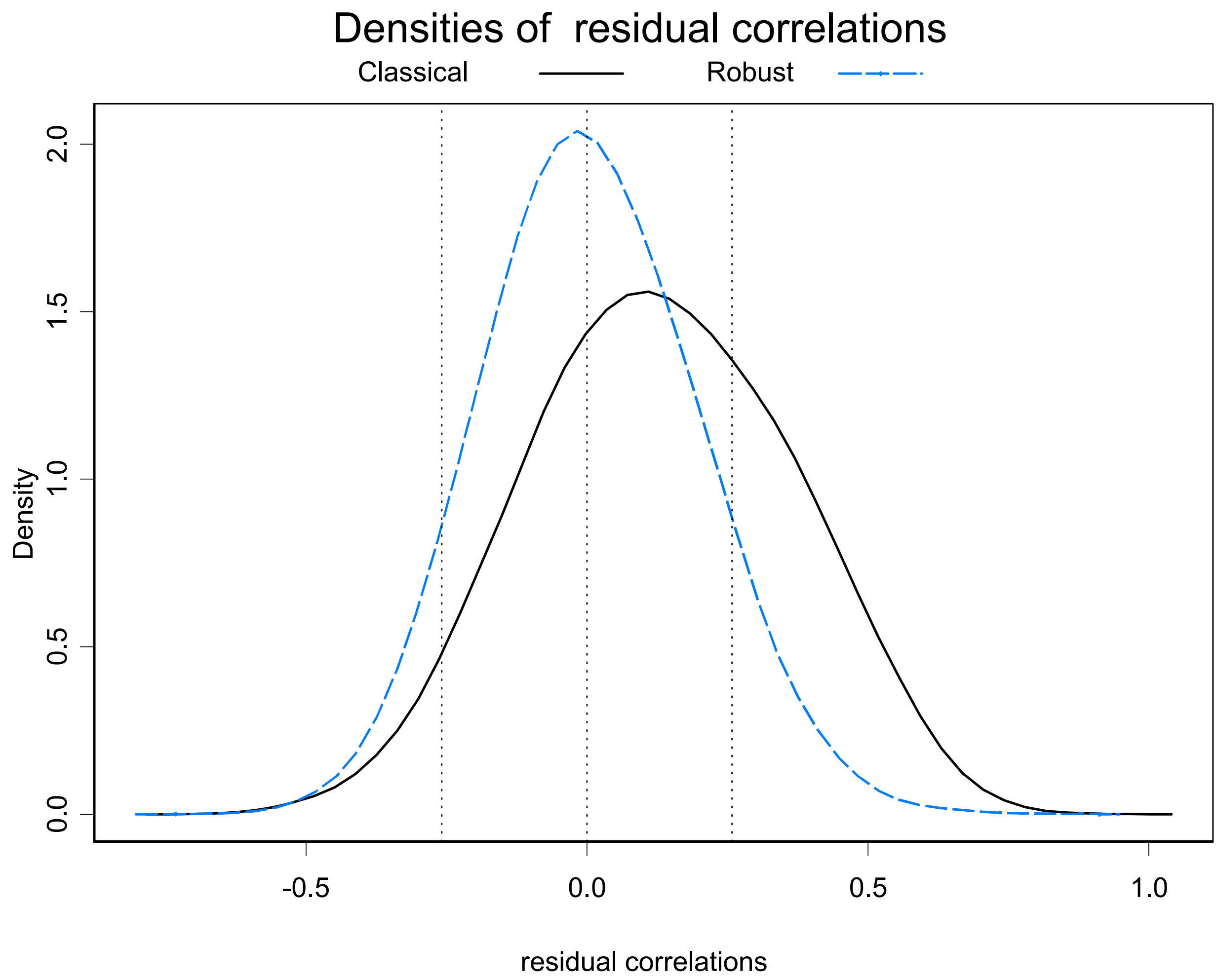
- R package now in “factorAnalytics”, originally created by Chris Green and others, with Doug Martin, and ported to R by Guy Yollin. Likely to see further development
- Example of use in next slides

Robust versus Classical Factor Returns

Three risk factors: size, E/P, B/M, monthly returns



Residuals Cross-Section Correlations



Robust FoF Factor Models

K funds and M_k risk factors for k-th fund:

$$r_{k,t} = \alpha_k + \beta_{k,1} f_{k,1,t} + \beta_{k,2} f_{k,2,t} + \ldots + \beta_{k,M_k} f_{k,M_k,t} + \varepsilon_{k,t}$$

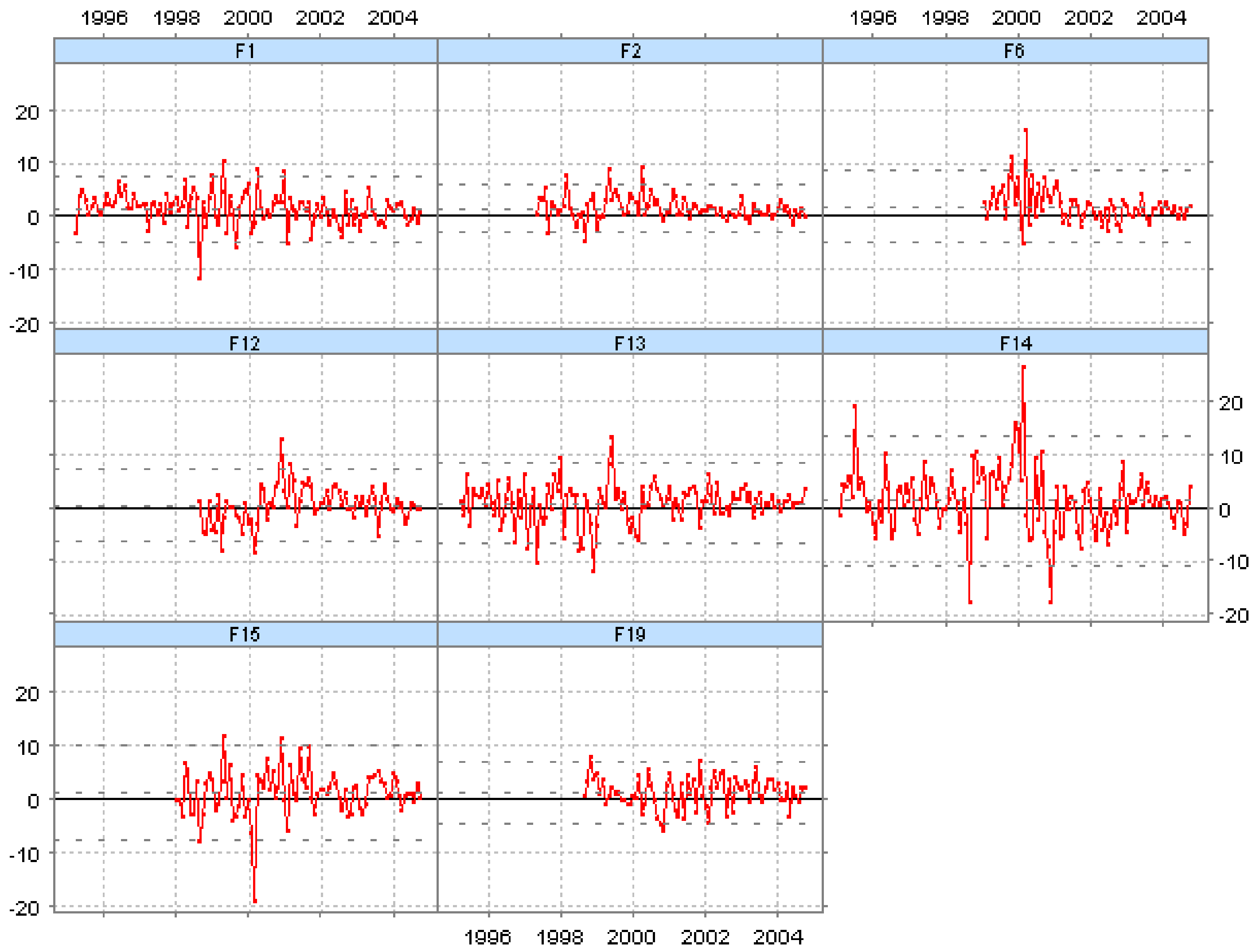
- Identify main risk factor drivers for each fund
- Create good fat-tailed skewed multivariate distribution simulation model
- Use model for full risk decomposition analysis, with manager groupings and risk factor groupings, and for portfolio optimization

Advantages of Robust TSFM Fits

- Reduced bias and variability of exposures estimates
- Smaller exposures estimate standard errors
- More accurate robust t-statistics
- Robust R-squared (not implemented in next examples)
- Robust model selection criterion
  - Needed for better model selection (OLS can give wrong model)
  - Robust F-test for comparing two models
  - Robust version (RFPE) of Akaike FPE

Reference for robustness details: Maronna, Martin and Yohai (2006)

FoF Portfolio Returns Example



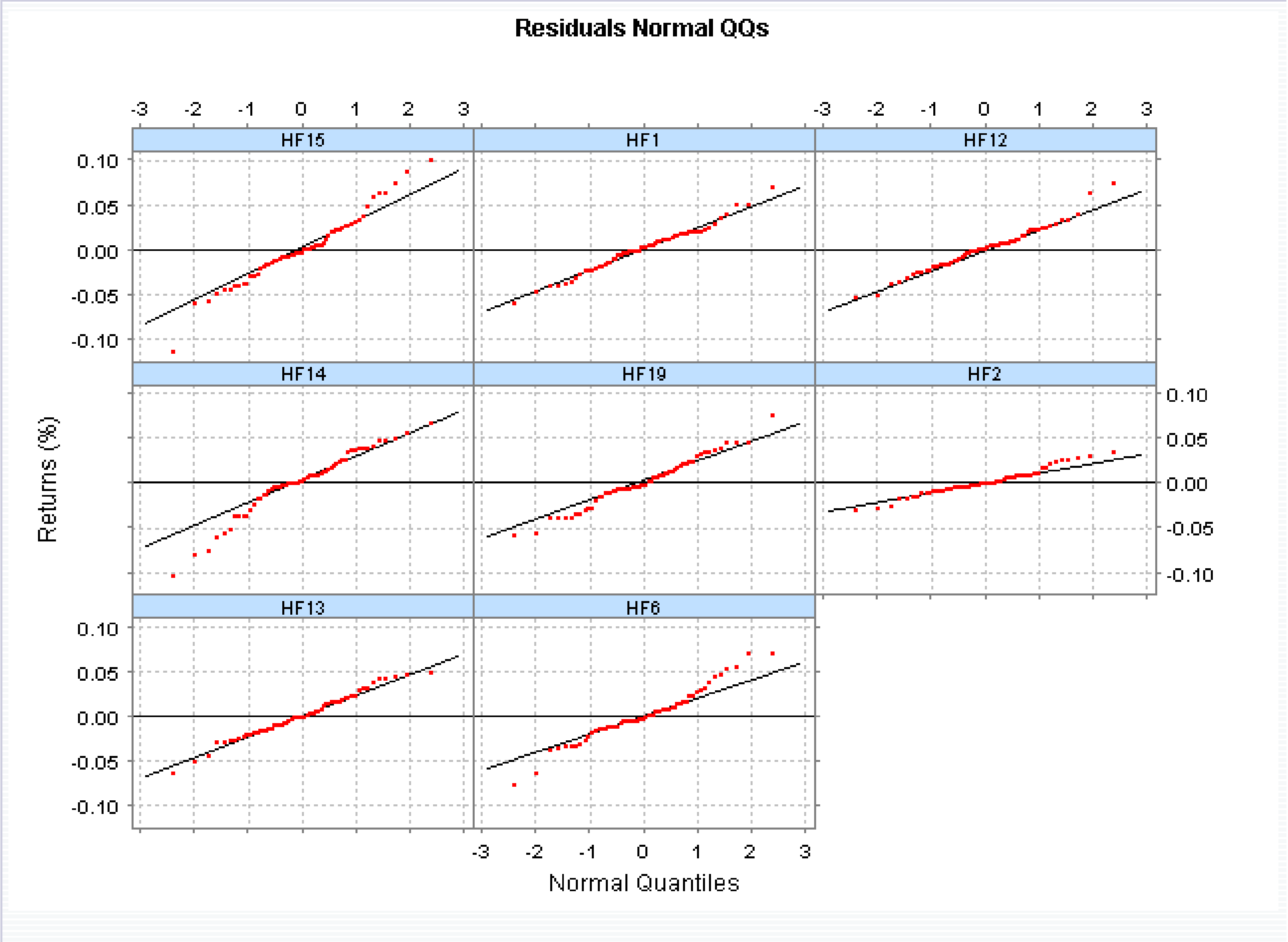
OLS Stepwise Model Selection

	Global Macro Index	Long/Short Equity Index	Hedge Fund Index	Emerging Markets Index	Distressed Index	Dedicated Short Bias Index	Event Driven Multi-Strategy Index	@Intercept	Multi-Strategy Index	Convertible Arbitrage Index	R Square	Residual Standard Error
HF15		**-0.719** 0.265 (-1.251, -0.188) 0.0067		**0.247** 0.255 (-0.265, 0.759) 0.334	**0.945** 0.432 (0.079, 1.810) 0.029			**0.016** 0.0067 (0.003, 0.030) 0.015		**-1.175** 0.455 (-2.086, -0.264) 0.0098	**0.252**	**0.04**
HF1				**0.279** 0.093 (0.093, 0.465) 0.0027				**0.0058** 0.0034 (-0.001, 0.013) 0.091			**0.135**	**0.026**
HF12	**0.998** 0.233 (0.532, 1.465) 0.000018		**-1.269** 0.329 (-1.929, -0.609) 0.00012		**0.816** 0.225 (0.365, 1.267) 0.00029	**0.219** 0.09 (0.038, 0.400) 0.015					**0.517**	**0.025**
HF14	**0.179** 0.59 (-1.003, 1.361) 0.762	**3.462** 1.073 (1.313, 5.611) 0.0012	**-4.287** 2.231 (-8.758, 0.184) 0.055			**-0.474** 0.128 (-0.730, -0.217) 0.00021				**1.297** 0.429 (0.438, 2.157) 0.0025	**0.72**	**0.036**
HF19									**1.201** 0.329 (0.543, 1.860) 0.00026		**0.184**	**0.029**
HF2		**-0.379** 0.097 (-0.572, -0.185) 0.00009		**0.511** 0.092 (0.328, 0.695) 0.000000023		**0.146** 0.05 (0.046, 0.247) 0.0034			**0.821** 0.297 (0.226, 1.415) 0.0057	**0.481** 0.182 (0.116, 0.846) 0.0083	**0.67**	**0.014**
HF13		**-1.013** 0.406 (-1.827, -0.200) 0.013	**1.315** 0.676 (-0.039, 2.669) 0.052					**0.0062** 0.0041 (-0.002, 0.014) 0.133			**0.141**	**0.026**
HF6								**0.012** 0.0051 (0.001, 0.022) 0.025		**0.625** 0.321 (-0.018, 1.268) 0.052	**0.061**	**0.032**

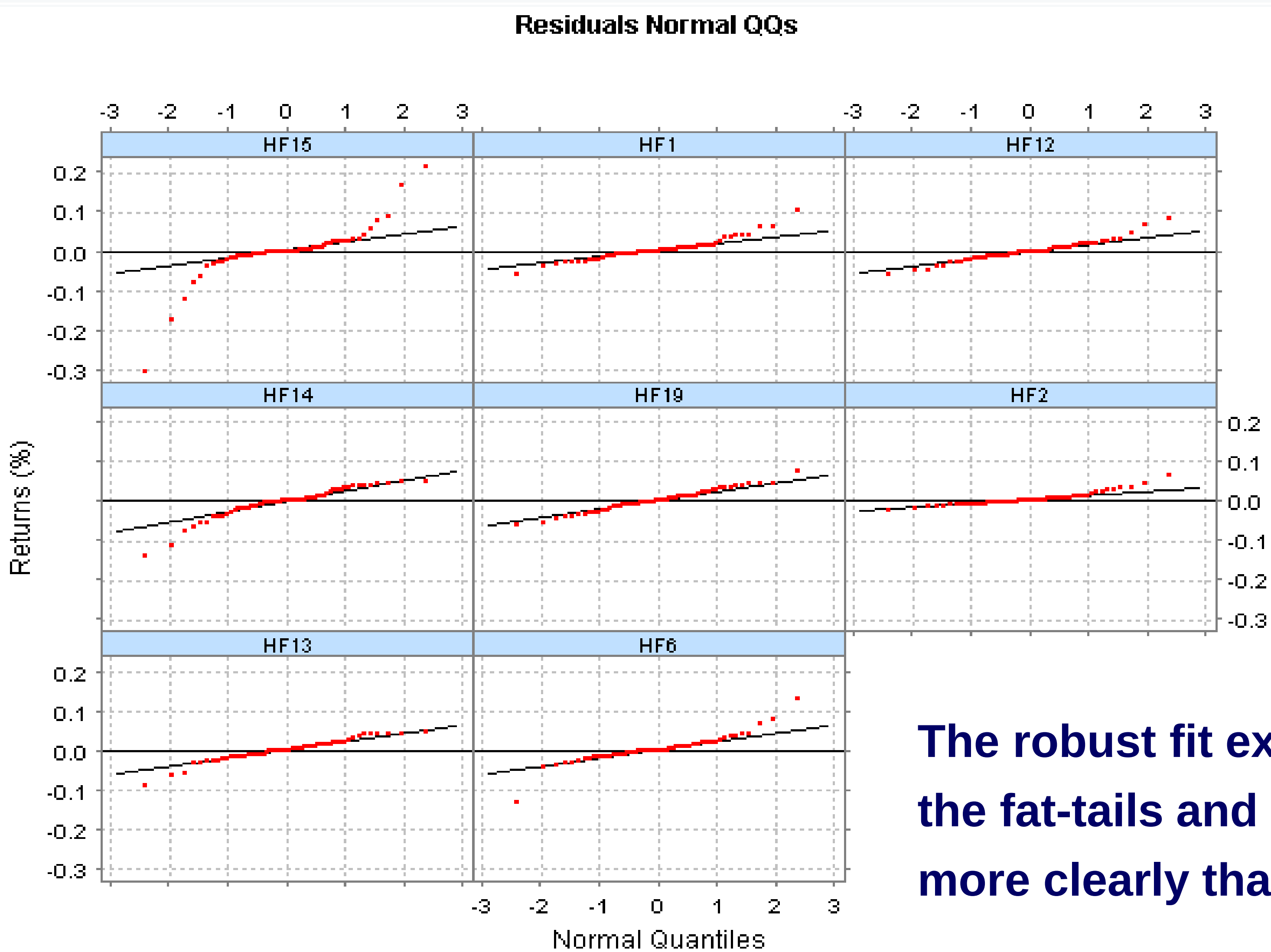
Robust Stepwise Model Selection

	Global Macro	Long/Short Equity	Hedge Fund	Emerging Markets	Distressed	Dedicated Short Bias	Event Driven Multi-Strat.	@Intercept	Multi-Strategy	Convertible Arbitrage	R Square	Residual Standard Error
HF15					**0.626** 0.284 (0.058, 1.195) 0.027			**0.011** 0.0051 (0.001, 0.022) 0.025		**-0.539** 0.362 (-1.264, 0.186) 0.137	**0.045**	**0.043**
HF1				**0.27** 0.095 (0.080, 0.460) 0.0044				**0.0051** 0.0035 (-0.002, 0.012) 0.147			**0.127**	**0.026**
HF12	**0.654** 0.16 (0.335, 0.974) 0.000042	**-0.591** 0.144 (-0.879, -0.304) 0.000038			**0.581** 0.193 (0.194, 0.967) 0.0026	**0.143** 0.081 (-0.019, 0.306) 0.077					**0.392**	**0.026**
HF14		**3.113** 0.605 (1.901, 4.325) 0.00000027	**-3.767** 0.963 (-5.697, -1.837) 0.000092			**-0.513** 0.116 (-0.745, -0.281) 0.0000092				**1.678** 0.361 (0.954, 2.402) 0.0000034	**0.752**	**0.037**
HF19						**-0.131** 0.081 (-0.292, 0.030) 0.104			**0.992** 0.347 (0.298, 1.687) 0.0043		**0.225**	**0.029**
HF2				**0.187** 0.061 (0.065, 0.310) 0.0021	**0.177** 0.134 (-0.091, 0.445) 0.186	**0.122** 0.048 (0.025, 0.218) 0.012		**0.0031** 0.0022 (-0.001, 0.008) 0.159	**0.479** 0.226 (0.025, 0.932) 0.034		**0.187**	**0.018**
HF13		**-0.294** 0.116 (-0.525, -0.062) 0.011			**0.397** 0.231 (-0.065, 0.859) 0.086			**0.0072** 0.0038 (-0.000, 0.015) 0.058			**0.118**	**0.026**
HF6	**0.844** 0.192 (0.460, 1.228) 0.00001		**-0.64** 0.452 (-1.544, 0.265) 0.157	**-0.288** 0.185 (-0.659, 0.082) 0.119	**-0.682** 0.315 (-1.313, -0.050) 0.031		**2.131** 0.53 (1.069, 3.192) 0.000057				**0.415**	**0.035**

Residuals QQ-Plots for OLS Fit



Residuals QQ-Plots for Robust Fit



The robust fit exposes the fat-tails and skew more clearly than OLS

5. ROBUST COVARIANCES

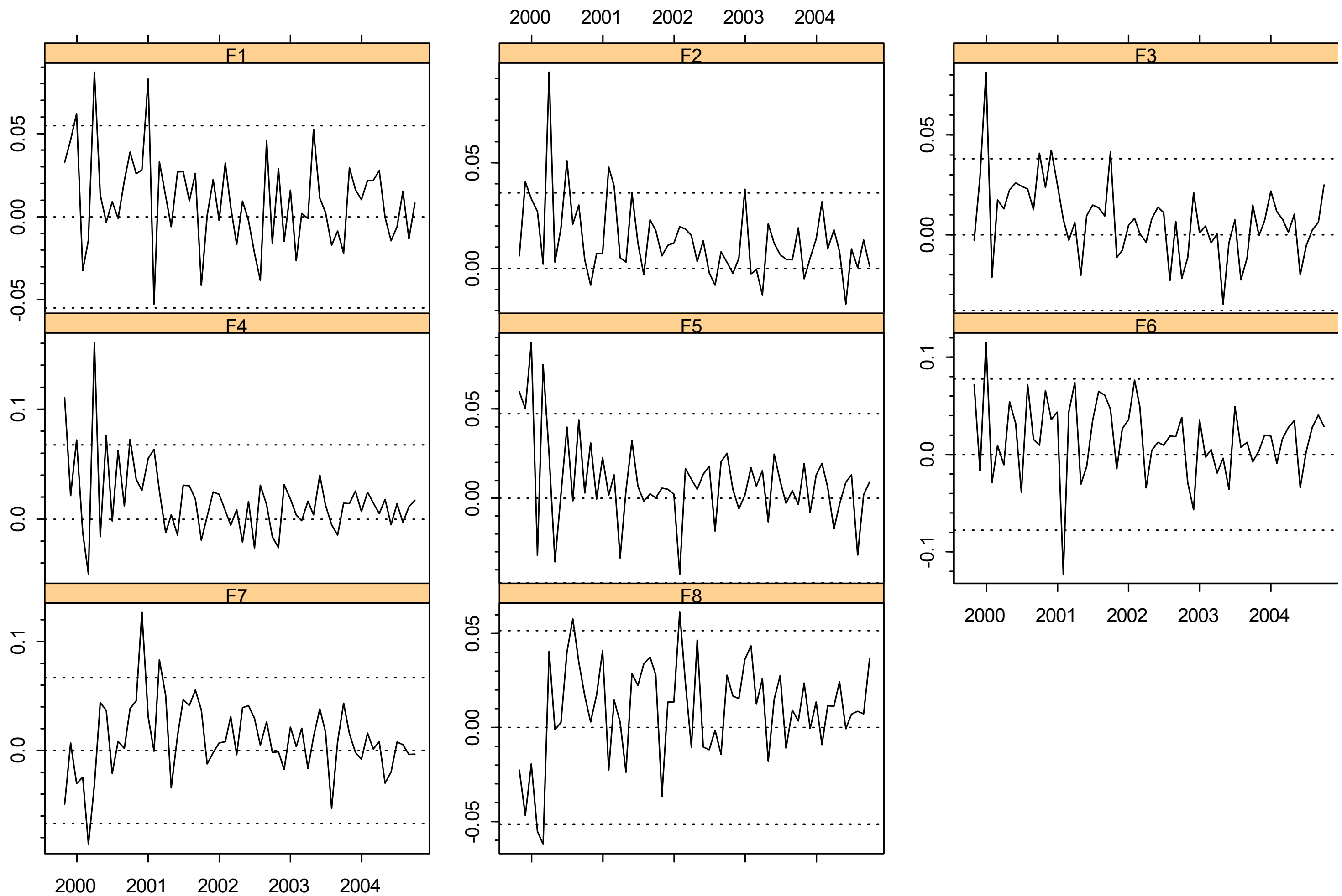
Uses in Portfolio Management

- Exploring asset returns correlations
- Detecting multi-dimensional outliers
- Robust mean-variance portfolio optimization

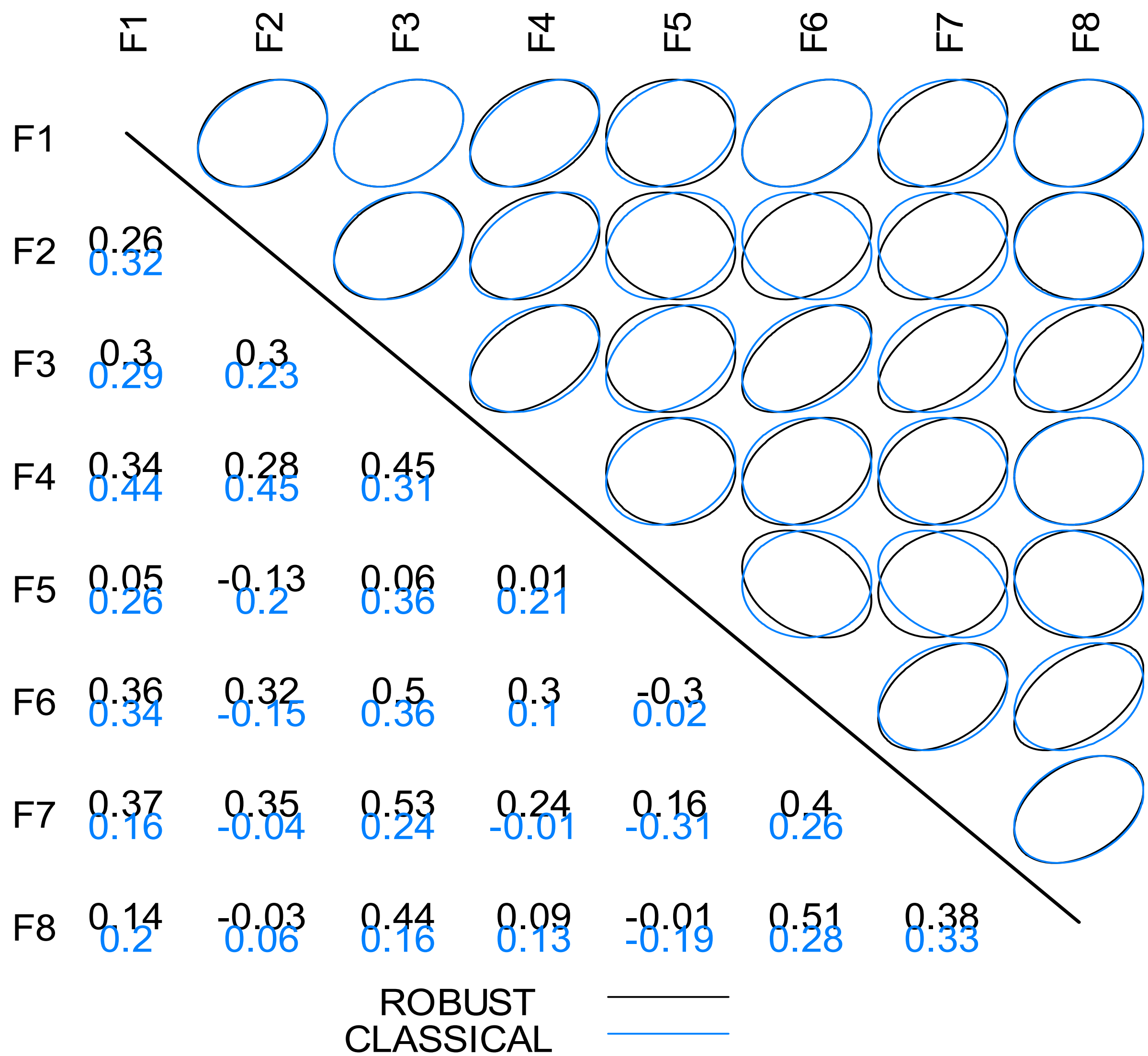
Types of Estimates

- M-estimates (Maronna, 1976)
- Min. covariance det. (MCD) (Rousseeuw, 197x)
- Pairwise estimates (Maronna & Zamar, 197x)

Hedge Fund Returns Example



Hedge Fund Returns Example



Portfolio Unusual Movement Alerts

Useful for portfolios with not too many assets, “general” fund-of-funds (FoF), including manager-of-managers portfolios.

- Retrospective guidance in allocation decisions
- Dynamic unusual movement detection
- Entire fund and style sub-groups

Robust Distances

So-called Mahalanobis distance

$$d_t^2 = \left(\mathbf{r}_t - \hat{\boldsymbol{\mu}}\right)' \hat{\boldsymbol{\Omega}}^{-1} \left(\mathbf{r}_t - \hat{\boldsymbol{\mu}}\right)$$
$$= \boldsymbol{z}_t' \boldsymbol{z}_t$$

Euclidean distance in new "spherized" coordinate system.

Classical version uses classical sample mean and sample covariance estimates. Replace them with highly robust versions, e.g., sample median and Fast Minimum Covariance Determinant (MCD) estimate.

Spherizing the Data

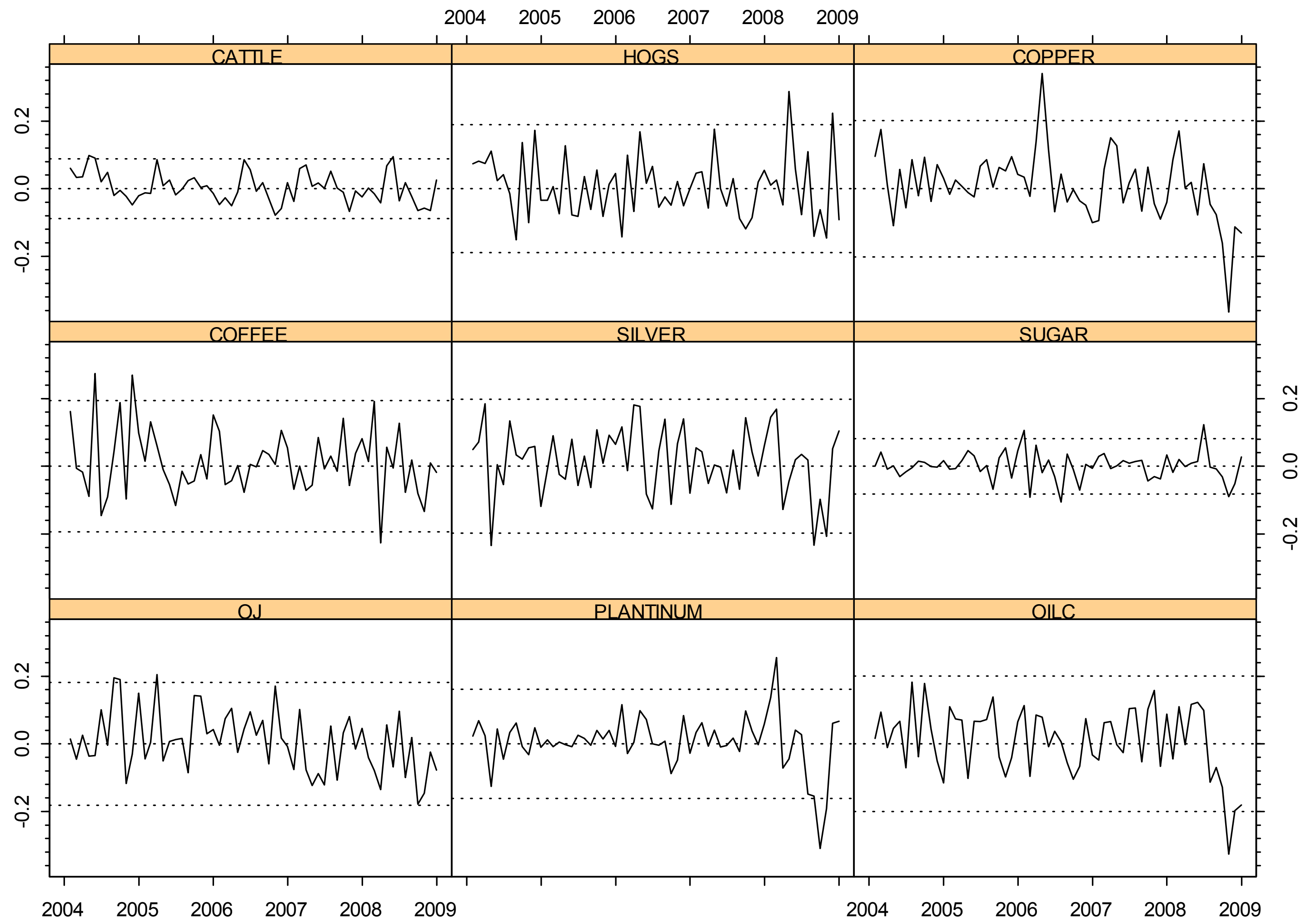
- If the $\mathbf{r}_t$ have an elliptical scatter, then the transformed variables

$$\mathbf{z}_t = \hat{\boldsymbol{\Omega}}^{-1/2}(\mathbf{r}_t - \overline{\mathbf{r}})$$

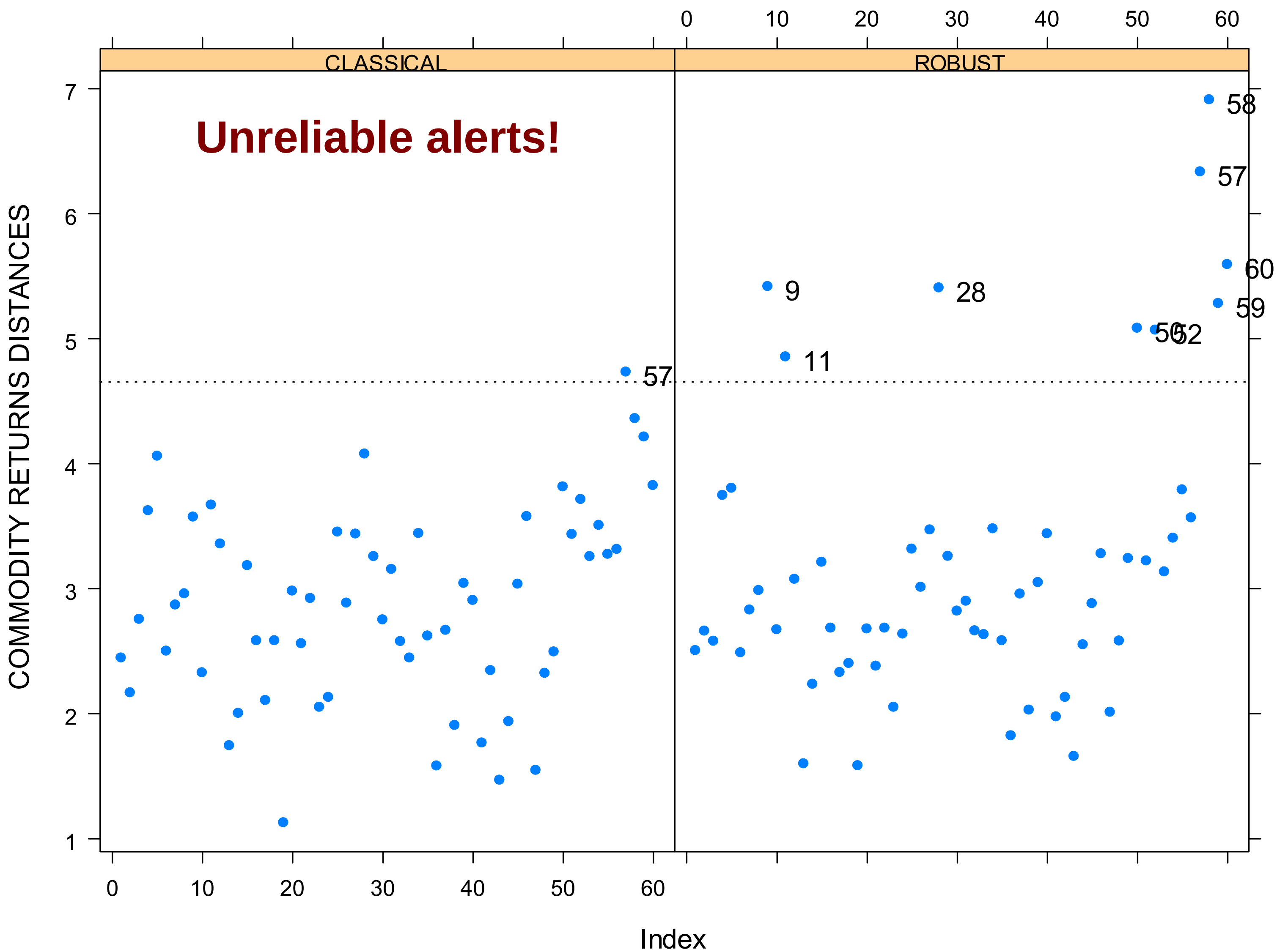
have a spherical scatter: $\operatorname{cov}(\mathbf{z}_t) = \mathbf{I}_p$

Commodities Example

(see Appendix A of Martin, Clark and Green, 2009)



Classical Alerts Robust Alerts



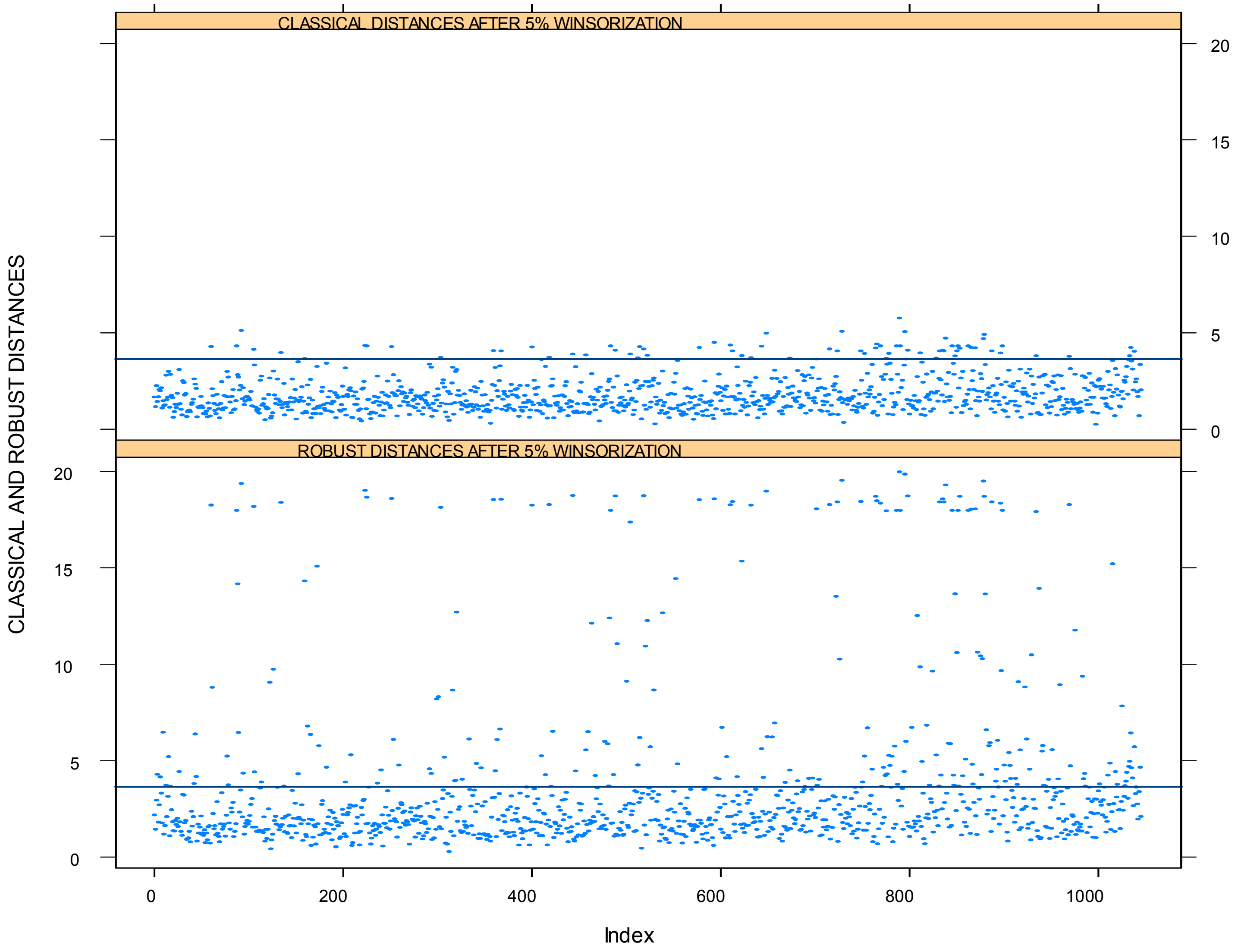
Fundamental Factors Multi-D Outliers

Martin, R. D., Clark, A and Green, C. G. (2010).

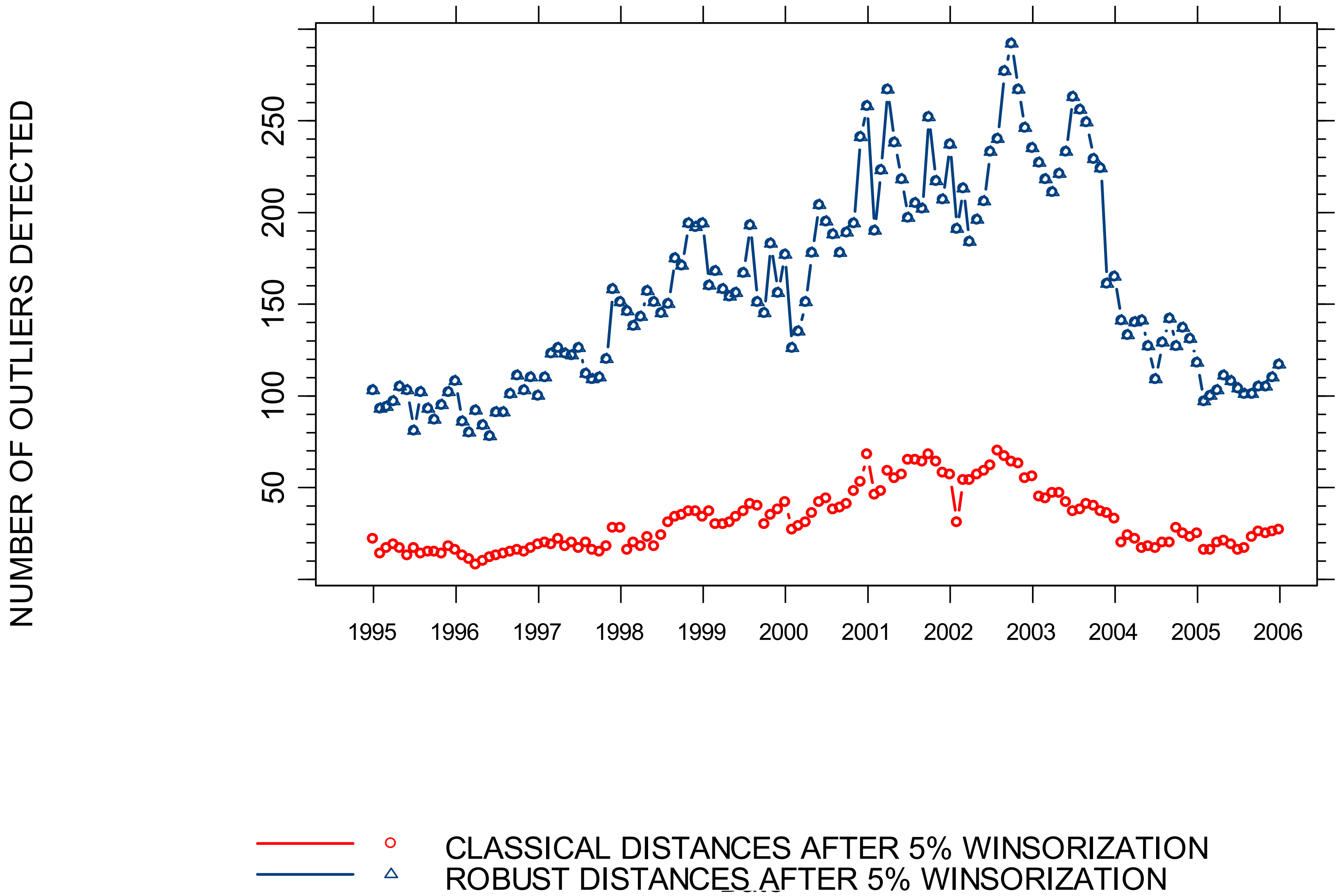
- Fundamental factor model context
- 4-D example: Size, B/M, E/P, Momentum
- Monthly data 1995-2006
- 1,046 equities

Robust vs Classical Outlier Detection

Month ending 7/31/2002



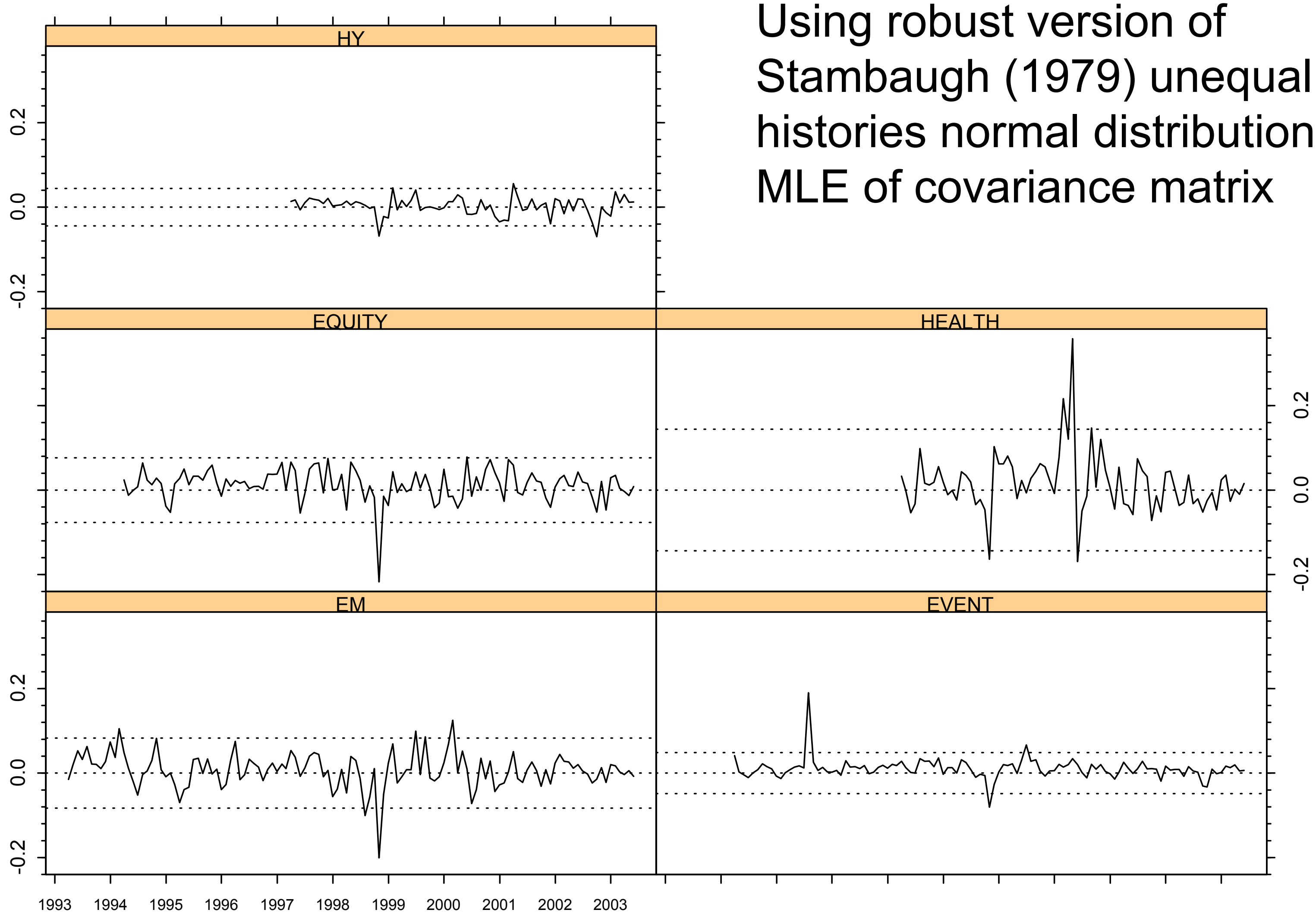
Number of 4-D Outliers Detected



Robust Mean-Variance Portfolios

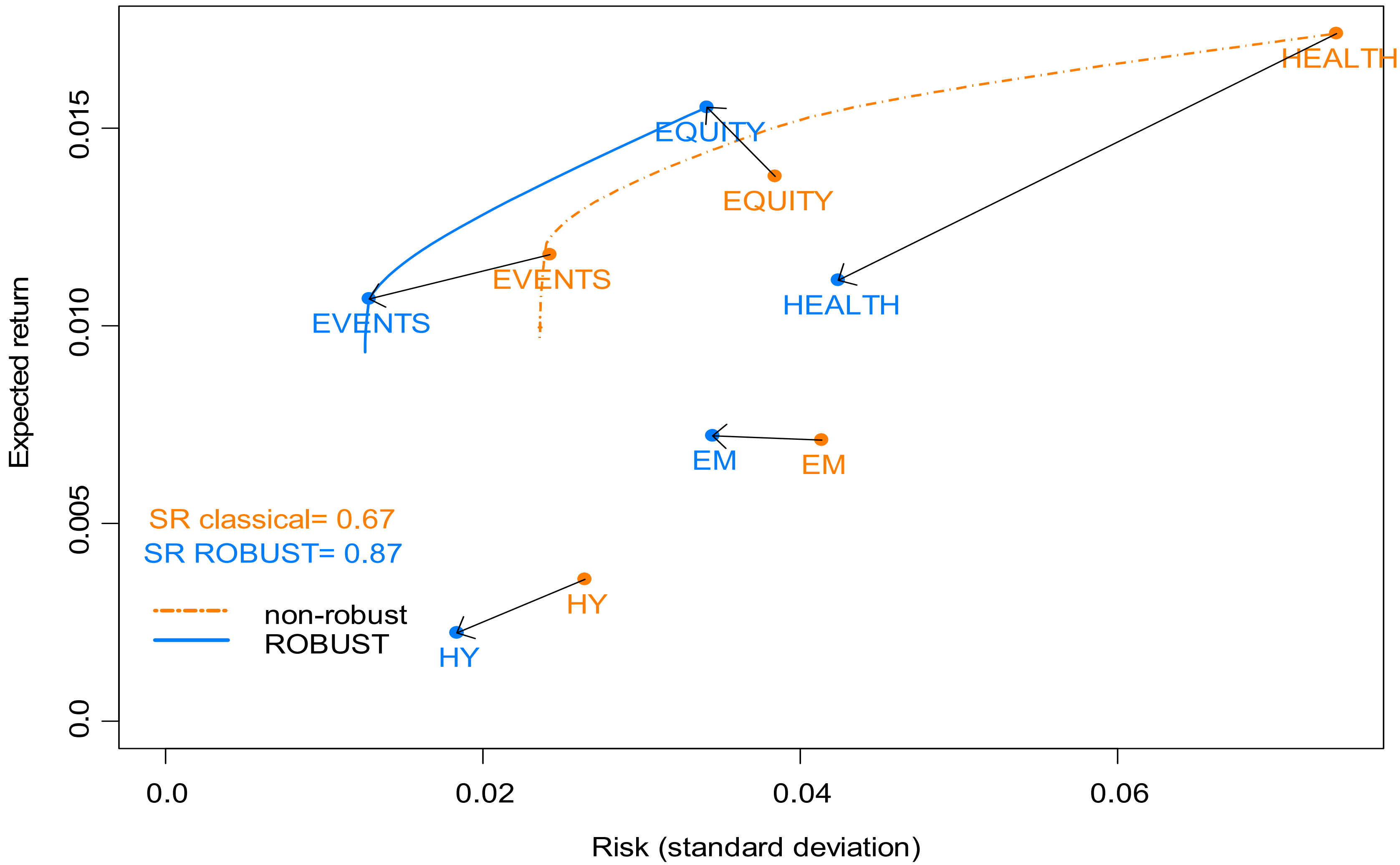
- **Use a Robust Covariance Matrix Estimate**
- **Mean Return Estimates of Your Choice**
  - Robust sample means
  - Robust alpha forecast, possibly Bayes
- **Primary Use**
  - Diagnostic: detect outliers influence
  - Examine returns, think hard, may choose robust MVO

FoHF Example 1



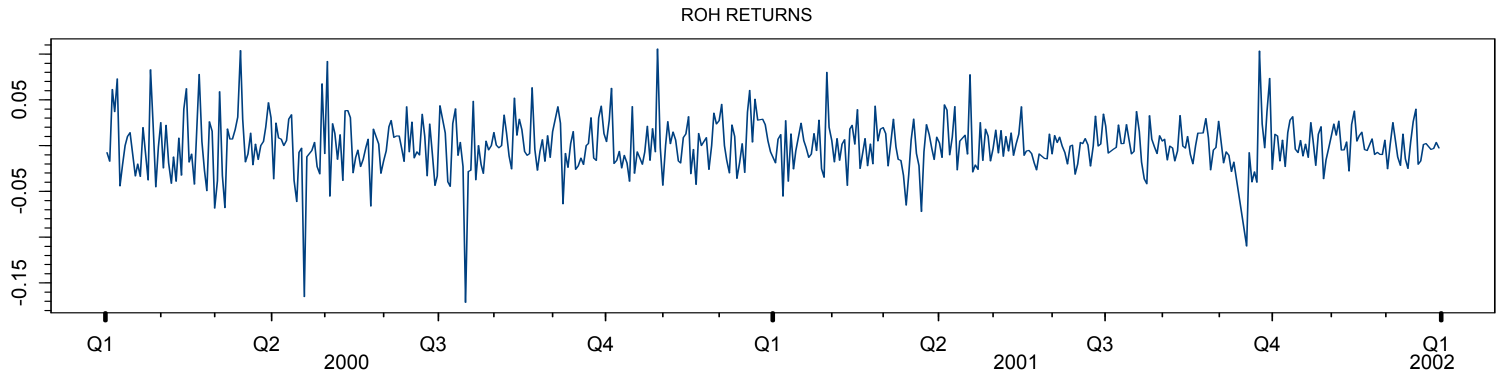
Using robust version of Stambaugh (1979) unequal histories normal distribution MLE of covariance matrix

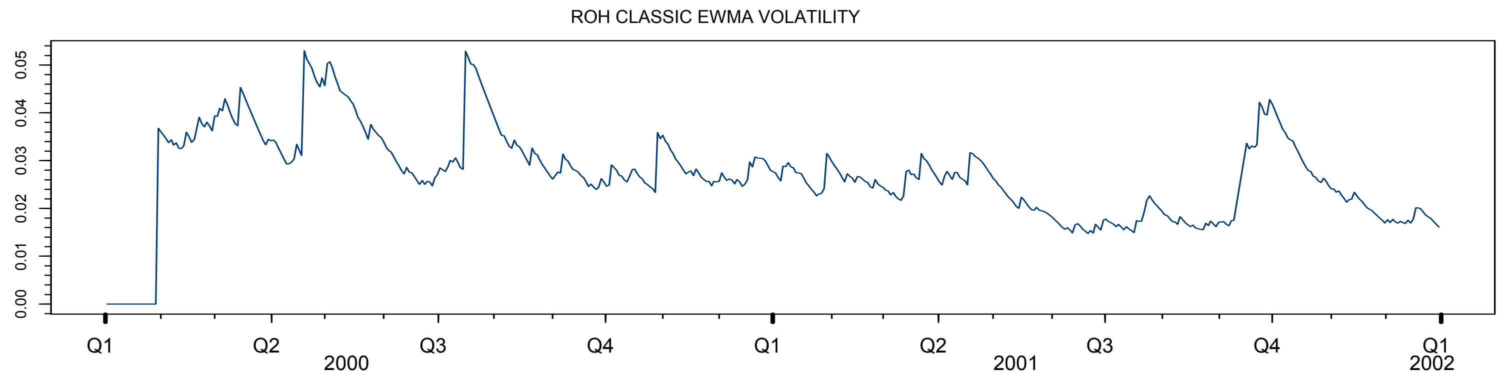
Which Do You Choose?



6. ROBUST VOLATILITY ESTIMATES

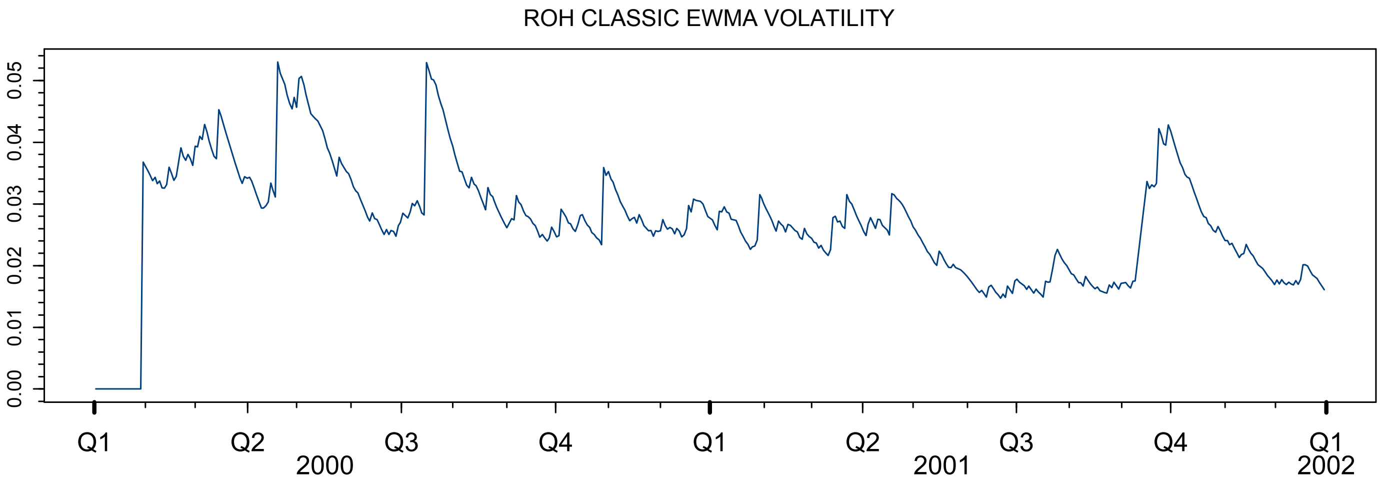
Classic EWMA: $\hat{\sigma}^2_{t+1} = \lambda \cdot \hat{\sigma}^2_t + (1-\lambda) \cdot r^2_{t+1}, \quad t \geq t_o$

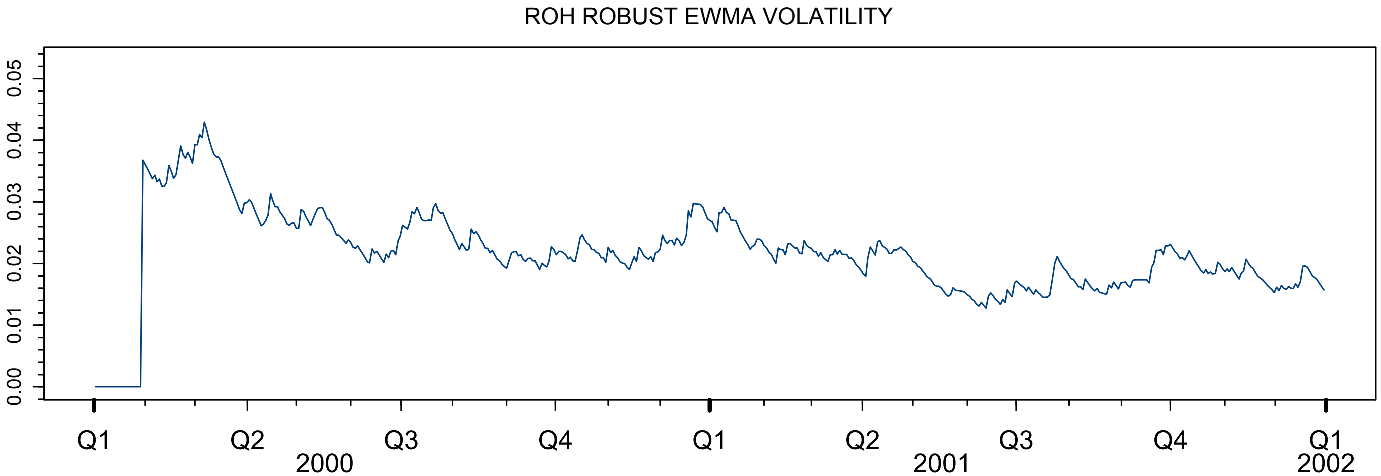




Over-estimates volatilities after outlier returns!

Robust EWMA Volatility Estimates





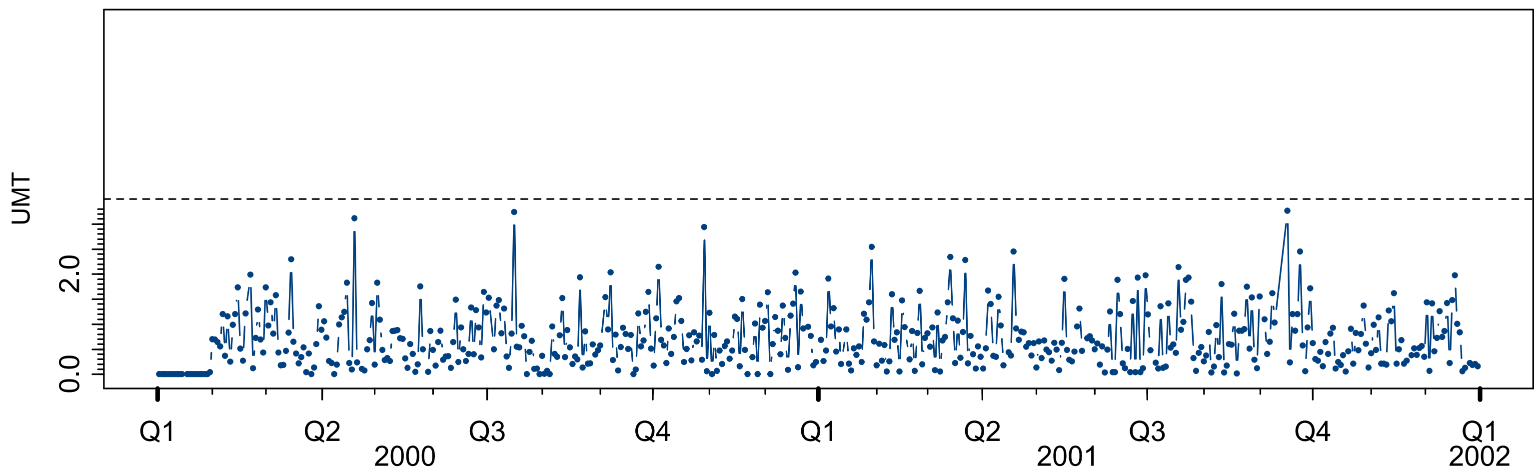
Robust EWMA

$$\begin{aligned} \hat{\sigma}_{t+1}^{2} &= \lambda \cdot \hat{\sigma}_{\mathrm{t}}^{2} + (1-\lambda) \cdot \mathrm{r}_{\mathrm{t+1}}^{2}, && \text{if} \quad \left|\mathrm{r}_{\mathrm{t+1}}\right| \leq a \cdot \hat{\sigma}_t \\ &= \hat{\sigma}_t^2, && \text{if} \quad \left|\mathrm{r}_{\mathrm{t+1}}\right| > a \cdot \hat{\sigma}_t \end{aligned}$$

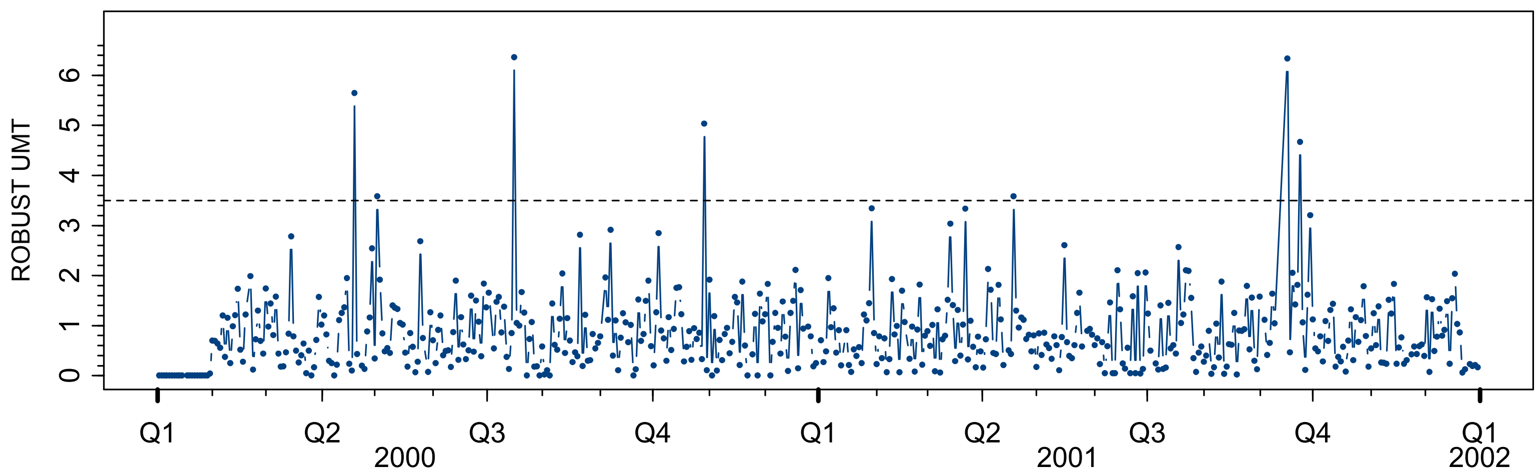
Unusual Movement Test Statistic

$$UMT_t = \frac{|r_t|}{\hat{\sigma}_t}$$

ROH UNUSUAL MOVEMENT TEST



ROH ROBUST UNUSUAL MOVEMENT TEST



Robust EWMA and GARCH References

- **Robust EWMA**
  - Scherer and Martin (2005), Section 6.4.2

- **Stable Distribution EWMA**
  - Stoyanov (2005)

$$\sigma_{t+1,t}^{p} = \lambda \sigma_{t,t-1}^{p} + \frac{(1-\lambda)}{C} |r_t|^p$$

$$C = E|r_t|^p$$

$$r_t \sim S_\alpha(\beta, 1, 0)$$

$$0 < p < \alpha$$

- **Robust GARCH**
  - Franses, van Djik and Lucas (1998)
  - Gregory and Reeves (2001)
  - Park (2002)
  - Muler and Yohai (2006)
  - Boudt and Croux (2006)

“Statistics is a science in my opinion, and it is no more a branch of mathematics than are physics, chemistry and economics; for if its methods fail the test of experience – not the test of logic – they will be discarded”

- J. W. Tukey

Thank You!

APPENDIX: R Robust Library Inference

- Standard errors, t-statistics, p-values, R^2
  - Asymptotically correct when there is no bias, and good approximations when bias is small

- Robust Test for Comparing Two Models
  - Uses the generic function **anova**
  - Default is **robust F-test**, alternative is **robust Wald test**

- Robust Model Selection
  - Robust version of Akaike FPE: **RFPE**
  - Used in backward stepwise selection

Robust Coefficient Covariance Matrix

$$\boxed{\boldsymbol{V} = \operatorname{cov}(\hat{\boldsymbol{\beta}}) = (\boldsymbol{X}^T \boldsymbol{X})^{-1} \cdot \mathrm{v}_{\mathrm{loc}}}$$

$$\mathrm{v}_{\mathrm{loc}} = \frac{s^2 \cdot E\psi^2\left(\frac{\boldsymbol{\varepsilon}}{s}\right)}{\left[E\psi'\left(\frac{\boldsymbol{\varepsilon}}{s}\right)\right]^2}$$

$$\boxed{\hat{\boldsymbol{V}} = (\tilde{\boldsymbol{X}}^T \tilde{\boldsymbol{X}})^{-1} \cdot \hat{\mathrm{v}}_{\mathrm{loc}}}$$

$$\tilde{\boldsymbol{X}} = \boldsymbol{W} \cdot \boldsymbol{X}$$

Diagonal matrix of weights
from final M-estimate.

Robust Standard Errors and t-Statistics

$$\boxed{s.e(\hat{\beta}_i) = \hat{V}_{ii}}$$

$$\boxed{t_i = \frac{\hat{\beta}_i}{s.e.(\hat{\beta}_i)}}$$

Robust R-Squared

$$R^2 = \frac{\sum_{i=1}^{n} \rho\left(\frac{y_i - \hat{\boldsymbol{\mu}}}{\hat{s}^o}\right) - \sum_{i=1}^{n} \rho\left(\frac{y_i - \boldsymbol{x}_i^T \boldsymbol{\beta}}{\hat{s}^o}\right)}{\sum_{i=1}^{n} \rho\left(\frac{y_i - \hat{\boldsymbol{\mu}}}{\hat{s}^o}\right)}$$

Reduces to classic R-squared when ρ is quadratic!

Robust F-Test

$p > q$

$$\boxed{F = 2 \cdot \frac{n-p}{p-q} \cdot \sum_{i=1}^{n} \left[\rho\left(\frac{y_i - \boldsymbol{x}_{q,i}^T \hat{\boldsymbol{\beta}}_q}{\hat{s}_p^o} \right) - \rho\left(\frac{y_i - \boldsymbol{x}_{p,i}^T \hat{\boldsymbol{\beta}}_p}{\hat{s}_p^o} \right) \right]}$$

Model Selection with RFPE

For a p-dimensional model for a subset of p predictor variables:

$$\boxed{RFPE = \sum_{i=1}^{n} \boldsymbol{\rho}\left(\frac{y_i - \boldsymbol{x}_{p,i}^T \hat{\boldsymbol{\beta}}_p}{\hat{s}_p^o}\right) + p \cdot \frac{\hat{A}}{\hat{B}}}$$

$$\hat{A} = \frac{1}{n} \cdot \sum_{i=1}^{n} \boldsymbol{\psi}^2\left(\frac{r_i}{\hat{s}^o}\right) \qquad \hat{B} = \frac{1}{n} \cdot \sum_{i=1}^{n} \boldsymbol{\psi}'\left(\frac{r_i}{\hat{s}^o}\right) \qquad r_i = y_i - \boldsymbol{x}_{p,i}^T \hat{\boldsymbol{\beta}}_p$$

Stepwise Variable Selection with RFPE

- Backward stepwise method
- Fit full model and get robust scale estimate and weights
- Use weights from full model to fit weighted least squares for each sub-model
- Use resulting sub-model beta's as initial estimate, along with robust scale from full model, to get M-estimate for sub-model
- Compute RFPE at each step, and eliminate a variable only if RFPE goes down
- Carried out with the Robust Library function `step`.

Robust Tests for Bias

Yohai, Stahel and Zamar (1991)

- **Compare LS and MM-Estimate**
  - If there is a significant difference, use MM-Estimate
- **Compare initial S-Estimate and final MM-Estimate**
  - If there is a significant difference, use S-estimate
  - This is a refinement that may not be used very often