Implied expected returns and the choice of a mean-variance efficient portfolio proxy

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work in progress; comments are welcome!

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Equity allocation in practice

Markowitz mean-variance approach is used in practice, but faces many problems/limitations:

- Optimization is subject to estimation risk when relying on past data

  $\Rightarrow$ Highly-concentrated and unstable portfolios

Possible solutions:

- $1/N$ 'rule of thumb'
- Risk-based portfolio allocation solutions
- Shrinkage or resampling approaches
- Constraints on the weights

  $\Rightarrow$ Still remains the needs to estimate expected returns for mean-variance optimization and alpha generation

Our contribution

- Propose a way to estimate expected returns based on a reverse-engineering approach (extension of Black and Litterman (1992))
- Compute the implied expected returns from several risk-based mean-variance efficient portfolios
- Exploit the fundamental relation between the expected returns, covariance matrix and the corresponding set of mean-variance efficient portfolios
- We find a statistically significant improvement in the out-of-sample Sharpe ratio of mean-variance efficient portfolios constructed with our approach compared with the standard use of implied expected returns from the market portfolio

Outline

Agenda

Notations

- Market with N risky securities
- Generic portfolio $(N \times 1)$ vector $\mathbf{w}$
- Expected arithmetic returns (in excess of the risk-free rate) at the desired holding horizon are denoted by the $(N \times 1)$ vector $\boldsymbol{\mu}$
- Corresponding $(N \times N)$ covariance matrix of arithmetic returns is denoted by $\boldsymbol{\Sigma}$
- We denote by $\boldsymbol{\iota}$ the $(N \times 1)$ vector of ones and by $\mathbf{0}$ the $(N \times 1)$ vector of zeros

Setup

Our analysis builds on the assumption of mean-variance preferences. Let $0 < \gamma < \infty$ be the risk aversion parameter. The mean-variance optimization problem is:

$$\mathbf{w}^* \equiv \underset{\mathbf{w} \in \mathcal{C}_{\mathsf{FI}}}{\operatorname{argmax}} \left\{ \boldsymbol{\mu}' \mathbf{w} - \tfrac{1}{2} \gamma \, \mathbf{w}' \boldsymbol{\Sigma} \mathbf{w} \right\} \,, \tag{1}$$

where $\mathcal{C}_{\mathsf{FI}} \equiv \{\mathbf{w} \in \mathbb{R}^N \,|\, \mathbf{w}'\boldsymbol{\iota} = 1\}$ is the full-investment constraint.

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Linear relationship

The Lagrangian corresponding to the problem in (1) is:

$$\mathcal{L}(\mathbf{w}, l) \equiv \mathbf{w}'\boldsymbol{\mu} - \tfrac{\gamma}{2}\,\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} - l(\mathbf{w}'\boldsymbol{\iota} - 1)\,,$$

with $l \in \mathbb{R}$. The corresponding first order conditions are:

$$\begin{aligned} \boldsymbol{\mu} - \gamma\boldsymbol{\Sigma}\mathbf{w} - l\boldsymbol{\iota} &= \mathbf{0} \\ \mathbf{w}'\boldsymbol{\iota} &= 1\,. \end{aligned} \tag{2}$$

From (2), we note the linear relationship between $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}\mathbf{w}$:

$$\boldsymbol{\mu} = l\boldsymbol{\iota} + \gamma\,\boldsymbol{\Sigma}\mathbf{w}\,. \tag{3}$$

Note that as γ is finite, (3) excludes the minimum variance portfolio.

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Implied expected returns

- Linear relationship in (3) is well known
- Has been used to compute the so-called implied expected returns, using the market portfolio as a proxy for a mean-variance efficient portfolio
- One of the first comprehensive treatments of this approach is Best and Grauer (1985) and Black and Litterman (1992)
- We advocate that the market portfolio is only one possible proxy for a mean-variance efficient portfolio, and that other proxies may lead to more accurate implied expected returns
- We rely on the risk literature to test alternative proxies

Linear regression and forecast

Replacing $\boldsymbol{\mu}$, $\boldsymbol{\Sigma}$ and $\mathbf{w}$ with possibly noisy proxies, denoted by $\hat{\boldsymbol{\mu}}$, $\hat{\boldsymbol{\Sigma}}$ and $\hat{\mathbf{w}}$, yields the following linear regression framework:

$$y_i = a + b\,x_i + \varepsilon_i\,,$$

where $y_i \equiv \hat{\mu}_i$, $x_i \equiv [\hat{\boldsymbol{\Sigma}}\hat{\mathbf{w}}]_i$, with a and b the regression parameters and ε_i an error term, whereby the regression is over the cross-section of securities $(i = 1, ..., N)$. Study whether the accuracy of the proxy $\hat{\boldsymbol{\mu}}$ can be improved by taking the constrained least squares fit of the regression and use the forecast:

$$\tilde{\boldsymbol{\mu}} \equiv \hat{a} + \hat{b}\,(\hat{\boldsymbol{\Sigma}}\hat{\mathbf{w}})\,.$$

Agenda

1. Implied expected returns
2. **Proxies**
3. Empirical analysis
4. Current research

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Market portfolio

- Under the CAPM assumptions, the market portfolio $\mathbf{w}_{\text{mkt}}$ has the mean-variance efficiency property
- Increasing body of literature has criticized the mean-variance efficiency of the market capitalization portfolio, and proposed alternatives that (under different assumptions) are mean–variance efficient

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Equally-weighted portfolio

- DeMiguel et al. (2009) show that the naive $1/N$ allocation rule outperforms several optimized portfolios
- This portfolio is mean–variance efficient when the expected returns $\boldsymbol{\mu}$ are proportional to the total risk $\boldsymbol{\Sigma\iota}$
- We denote this portfolio by $\mathbf{w}_{\text{ew}}$

Equal-risk-contribution portfolio

- For a portfolio $\mathbf{w}$, the percentage volatility risk contribution of the ith asset in the portfolio is given by:

$$\%RC_i \equiv \frac{w_i[\mathbf{\Sigma w}]_i}{\mathbf{w}'\mathbf{\Sigma w}} \,.$$

- The equal-risk-contribution portfolio is the portfolio for which all assets contribute equally to the overall risk of the portfolio:

$$\mathbf{w}_{\text{erc}} \equiv \underset{\mathbf{w}\in\mathcal{C}_{\text{FI}}}{\operatorname{argmin}} \left\{ \sum_{i=1}^{N} (\%RC_i - \tfrac{1}{N})^2 \right\} \,.$$

- The equal-risk-contribution portfolio is mean-variance efficient under some assumptions

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Maximum diversification portfolio

- Choueifaty and Coignard (2008) define the *portfolio's diversification ratio* the portfolio with the maximum diversification ratio:

$$DR(\mathbf{w}) \equiv \frac{\mathbf{w}'\boldsymbol{\sigma}}{\sqrt{\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}}} \geq 1\,,$$

  where $\boldsymbol{\sigma} \equiv \sqrt{\text{diag}(\boldsymbol{\Sigma})}$ denotes the $(N \times 1)$ vector of standard deviations.
- When expected returns are proportional to their volatility, the maximum diversification portfolio coincides with the maximum Sharpe ratio portfolio
- We denote this portfolio by $\mathbf{w}_{\text{md}}$

Risk-efficient portfolio

- Amenc et al. (2011) recommend to construct a maximum Sharpe portfolio under the assumption that the stock's expected return is a deterministic function of its semi-deviation and the cross-sectional distribution of semi-deviations
- They sort stocks by their semi-deviation, form decile portfolios and then compute the median semi-deviation of stocks in each decile portfolio: ξ_j $(j = 1, \ldots, 10)$. The so-called risk–efficient portfolio is given by:

$$\mathbf{w}_{\text{ref}} \equiv \underset{\mathbf{w} \in \mathcal{C}_{\text{FI}}}{\operatorname{argmax}} \left\{ \frac{\mathbf{w}' \boldsymbol{J} \boldsymbol{\xi}}{\sqrt{\mathbf{w}' \boldsymbol{\Sigma} \mathbf{w}}} \right\} ,$$

where $\boldsymbol{J}$ is a $(N \times 10)$ matrix of zeros whose (i, j)-th element is one if the semi-deviation of stock i belongs to decile j, and $\boldsymbol{\xi} \equiv (\xi_1, \ldots, \xi_{10})'$.

Agenda

Portfolios

- We distinguish between the *return–insensitive* and the *return–sensitive* portfolios
- The return–sensitive portfolios that we consider are the solutions to the mean-variance optimization (1) with risk aversion level γ
- We follow Das et al. (2010) in calibrating γ at 0.8773 ("high risk portfolio"), 2.7063 ("medium risk") and 3.795 ("low risk")
- We consider a long-only portfolio and a 130-30 portfolio using the approach by Fan et al. (2009)

Setup

- Daily adjusted prices of the S&P 100 equities over the period January 1999 to December 2011
- Market capitalization at the end of each month
- Risk-free rate is the three-month Treasury bill
- All figures are in USD
- Monthly rebalancing frequency but rely on weekly prices to compute the various estimators, using a rolling window of three years, which is common practice in the financial industry
- The backtest period ranges from January 2002 to December 2011, for a total of 119 monthly observations

Return–insensitive portfolios

	$\mathbf{w}_{\text{mkt}}$	$\mathbf{w}_{\text{ew}}$	$\mathbf{w}_{\text{erc}}$	$\mathbf{w}_{\text{ref}}$	$\mathbf{w}_{\text{md}}$	$\mathbf{w}_{\text{min}}$
Mean	0.029	0.037	0.040	0.028	0.033	0.065
Vol.	0.153	0.162**	0.142**	0.130	0.131*	0.113***
Sharpe	0.193	0.230	0.282	0.213	0.253	0.580

Table: Annualized figures. ***, ** and * indicate significant differences between the portfolio considered and the market capitalization weighted portfolio at the 1%, 5% and 10% level, respectively. $\mathbf{w}_{\text{mkt}}$: market capitalization weighted portfolio; $\mathbf{w}_{\text{ew}}$: equally-weighted portfolio; $\mathbf{w}_{\text{erc}}$: equal-risk-contribution portfolio; $\mathbf{w}_{\text{ref}}$: risk-efficient portfolio; $\mathbf{w}_{\text{md}}$: maximum diversification portfolio; $\mathbf{w}_{\text{min}}$: minimum volatility portfolio.

Return–sensitive portfolios for $\gamma = 2.7063$ ("medium risk")

Long-only constraint: $c = 1$						
	ir-$\mathbf{w}_{\text{mkt}}$	ir-$\mathbf{w}_{\text{ew}}$	ir-$\mathbf{w}_{\text{erc}}$	ir-$\mathbf{w}_{\text{ref}}$	ir-$\mathbf{w}_{\text{md}}$	sample
Mean	0.027	0.043*	0.045*	0.044	0.041	0.009
Vol.	0.239	0.240	0.235	0.185	0.221	0.388*
Sharpe	0.114	0.181*	0.190**	0.240	0.184	0.023
Gross exposure constraint: $c = 1.6$						
	ir-$\mathbf{w}_{\text{mkt}}$	ir-$\mathbf{w}_{\text{ew}}$	ir-$\mathbf{w}_{\text{erc}}$	ir-$\mathbf{w}_{\text{ref}}$	ir-$\mathbf{w}_{\text{md}}$	sample
Mean	0.024	0.044**	0.044**	0.057	0.030	−0.013
Vol.	0.262	0.264	0.258	0.199	0.252	0.432**
Sharpe	0.093	0.165**	0.170**	0.288	0.120	−0.030

Table: Annualized figures. Long only $(c = 1)$ and 130%-30% gross constraints $(c = 1.6)$ portfolios. ***, ** and * indicate significant differences between the portfolio considered and the market capitalization implied expected return prediction portfolio at the 1%, 5% and 10% level, respectively. Sample denotes results for the naive past return estimation while ir- denotes implied returns obtained for the various proxy portfolios. $\mathbf{w}_{\text{mkt}}$: market capitalization weighted portfolio; $\mathbf{w}_{\text{ew}}$: equally-weighted portfolio; $\mathbf{w}_{\text{erc}}$: equal-risk-contribution portfolio; $\mathbf{w}_{\text{ref}}$: risk-efficient portfolio; $\mathbf{w}_{\text{md}}$: maximum diversification portfolio; $\mathbf{w}_{\text{min}}$: minimum volatility portfolio.

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Current research

Has been done:

- Evolution of the slope parameter
- Cross-sectional distribution of implied expected returns over time
- Four-factor regression analysis

To do:

- Sub-window performance analysis
- Expected return performance vs. shrinkage effects
- Alternative covariance matrix estimators
- Dynamically switching between methods
- R package

Thanks for your attention

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