Implied expected returns and the choice of a mean-variance efficient portfolio proxy

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joint work with Kris Boudt‡
work in progress; comments are welcome!

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Equity allocation in practice

Markowitz mean-variance approach is used in practice, but faces many problems/limitations:

- Optimization is subject to estimation risk when relying on past data
  ⇒ Highly-concentrated and unstable portfolios

Possible solutions:

- $1/N$ ‘rule of thumb’
- Risk-based portfolio allocation solutions
- Shrinkage or resampling approaches
- Constraints on the weights
  ⇒ Still remains the needs to estimate expected returns for mean-variance optimization and alpha generation
Our contribution

- Propose a way to estimate expected returns based on a reverse-engineering approach (extension of Black and Litterman (1992))
- Compute the implied expected returns from several risk-based mean-variance efficient portfolios
- Exploit the fundamental relation between the expected returns, covariance matrix and the corresponding set of mean-variance efficient portfolios
- We find a statistically significant improvement in the out-of-sample Sharpe ratio of mean-variance efficient portfolios constructed with our approach compared with the standard use of implied expected returns from the market portfolio
Outline

1. Implied expected returns
2. Proxies
3. Empirical analysis
4. Current research
Agenda

1. Implied expected returns
2. Proxies
3. Empirical analysis
4. Current research
Notations

- Market with $N$ risky securities
- Generic portfolio $(N \times 1)$ vector $w$
- Expected arithmetic returns (in excess of the risk-free rate) at the desired holding horizon are denoted by the $(N \times 1)$ vector $\mu$
- Corresponding $(N \times N)$ covariance matrix of arithmetic returns is denoted by $\Sigma$
- We denote by $\iota$ the $(N \times 1)$ vector of ones and by $0$ the $(N \times 1)$ vector of zeros
Our analysis builds on the assumption of mean-variance preferences. Let $0 < \gamma < \infty$ be the risk aversion parameter. The mean-variance optimization problem is:

$$
\mathbf{w}^* \equiv \arg\max_{\mathbf{w} \in C_{FI}} \left\{ \mu' \mathbf{w} - \frac{1}{2} \gamma \mathbf{w}' \Sigma \mathbf{w} \right\},
$$

where $C_{FI} \equiv \{ \mathbf{w} \in \mathbb{R}^N \mid \mathbf{w}' \boldsymbol{\iota} = 1 \}$ is the full-investment constraint.
Linear relationship

The Lagrangian corresponding to the problem in (1) is:

\[ \mathcal{L}(w, l) \equiv w' \mu - \frac{\gamma}{2} w' \Sigma w - l(w' \iota - 1), \]

with \( l \in \mathbb{R} \). The corresponding first order conditions are:

\[ \mu - \gamma \Sigma w - l \iota = 0 \]
\[ w' \iota = 1. \] 

From (2), we note the linear relationship between \( \mu \) and \( \Sigma w \):

\[ \mu = l \iota + \gamma \Sigma w. \] 

Note that as \( \gamma \) is finite, (3) excludes the minimum variance portfolio.
Implied expected returns

- Linear relationship in (3) is well known
- Has been used to compute the so-called implied expected returns, using the market portfolio as a proxy for a mean-variance efficient portfolio
- One of the first comprehensive treatments of this approach is Best and Grauer (1985) and Black and Litterman (1992)
- We advocate that the market portfolio is only one possible proxy for a mean-variance efficient portfolio, and that other proxies may lead to more accurate implied expected returns
- We rely on the risk literature to test alternative proxies
Replacing $\mu$, $\Sigma$ and $w$ with possibly noisy proxies, denoted by $\hat{\mu}$, $\hat{\Sigma}$ and $\hat{w}$, yields the following linear regression framework:

$$y_i = a + b x_i + \varepsilon_i,$$

where $y_i \equiv \hat{\mu}_i$, $x_i \equiv [\hat{\Sigma} \hat{w}]_i$, with $a$ and $b$ the regression parameters and $\varepsilon_i$ an error term, whereby the regression is over the cross-section of securities ($i = 1, ..., N$). Study whether the accuracy of the proxy $\hat{\mu}$ can be improved by taking the constrained least squares fit of the regression and use the forecast:

$$\tilde{\mu} \equiv \hat{a} + \hat{b} (\hat{\Sigma} \hat{w}).$$
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Market portfolio

- Under the CAPM assumptions, the market portfolio $\mathbf{w}_{mkt}$ has the mean-variance efficiency property.
- Increasing body of literature has criticized the mean-variance efficiency of the market capitalization portfolio, and proposed alternatives that (under different assumptions) are mean–variance efficient.
• DeMiguel et al. (2009) show that the naive $1/N$ allocation rule outperforms several optimized portfolios
• This portfolio is mean–variance efficient when the expected returns $\mu$ are proportional to the total risk $\Sigma \nu$
• We denote this portfolio by $w_{ew}$
Equal-risk-contribution portfolio

- For a portfolio $\mathbf{w}$, the percentage volatility risk contribution of the $i$th asset in the portfolio is given by:

$$\%RC_i \equiv \frac{w_i[\Sigma \mathbf{w}]_i}{\mathbf{w}'\Sigma \mathbf{w}}.$$  

- The equal-risk-contribution portfolio is the portfolio for which all assets contribute equally to the overall risk of the portfolio:

$$\mathbf{w}_{erc} \equiv \arg\min_{\mathbf{w} \in C_{FI}} \left\{ \sum_{i=1}^{N} (\%RC_i - \frac{1}{N})^2 \right\}.$$  

- The equal-risk-contribution portfolio is mean-variance efficient under some assumptions.
Choueifaty and Coignard (2008) define the portfolio’s diversification ratio as the portfolio with the maximum diversification ratio:

$$DR(w) \equiv \frac{w'\sigma}{\sqrt{w'\Sigma w}} \geq 1,$$

where $\sigma \equiv \sqrt{\text{diag}(\Sigma)}$ denotes the $(N \times 1)$ vector of standard deviations.

- When expected returns are proportional to their volatility, the maximum diversification portfolio coincides with the maximum Sharpe ratio portfolio.
- We denote this portfolio by $w_{md}$. 
• Amenc et al. (2011) recommend to construct a maximum Sharpe portfolio under the assumption that the stock’s expected return is a deterministic function of its semi-deviation and the cross-sectional distribution of semi-deviations
• They sort stocks by their semi-deviation, form decile portfolios and then compute the median semi-deviation of stocks in each decile portfolio: \( \xi_j \ (j = 1, \ldots, 10) \). The so-called risk–efficient portfolio is given by:

\[
\mathbf{w}_{\text{ref}} \equiv \arg\max_{\mathbf{w} \in \mathcal{C}_{\text{FI}}} \left\{ \frac{\mathbf{w}' \mathbf{J} \xi}{\sqrt{\mathbf{w}' \mathbf{\Sigma} \mathbf{w}}} \right\},
\]

where \( \mathbf{J} \) is a \((N \times 10)\) matrix of zeros whose \((i,j)\)-th element is one if the semi-deviation of stock \( i \) belongs to decile \( j \), and \( \xi \equiv (\xi_1, \ldots, \xi_{10})' \).
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Portfolios

- We distinguish between the *return–insensitive* and the *return–sensitive* portfolios
- The return–sensitive portfolios that we consider are the solutions to the mean-variance optimization (1) with risk aversion level $\gamma$
- We follow Das et al. (2010) in calibrating $\gamma$ at 0.8773 (“high risk portfolio”), 2.7063 (“medium risk”) and 3.795 (“low risk”)
- We consider a long-only portfolio and a 130-30 portfolio using the approach by Fan et al. (2009)
Setup

- Daily adjusted prices of the S&P 100 equities over the period January 1999 to December 2011
- Market capitalization at the end of each month
- Risk-free rate is the three-month Treasury bill
- All figures are in USD
- Monthly rebalancing frequency but rely on weekly prices to compute the various estimators, using a rolling window of three years, which is common practice in the financial industry
- The backtest period ranges from January 2002 to December 2011, for a total of 119 monthly observations
Return–insensitive portfolios

<table>
<thead>
<tr>
<th></th>
<th>$w_{\text{mkt}}$</th>
<th>$w_{\text{ew}}$</th>
<th>$w_{\text{erc}}$</th>
<th>$w_{\text{ref}}$</th>
<th>$w_{\text{md}}$</th>
<th>$w_{\text{min}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.029</td>
<td>0.037</td>
<td>0.040</td>
<td>0.028</td>
<td>0.033</td>
<td>0.065</td>
</tr>
<tr>
<td>Vol.</td>
<td>0.153</td>
<td>0.162**</td>
<td>0.142**</td>
<td>0.130</td>
<td>0.131*</td>
<td>0.113***</td>
</tr>
<tr>
<td>Sharpe</td>
<td>0.193</td>
<td>0.230</td>
<td>0.282</td>
<td>0.213</td>
<td>0.253</td>
<td>0.580</td>
</tr>
</tbody>
</table>

Table: Annualized figures. ***, ** and * indicate significant differences between the portfolio considered and the market capitalization weighted portfolio at the 1%, 5% and 10% level, respectively. $w_{\text{mkt}}$: market capitalization weighted portfolio; $w_{\text{ew}}$: equally-weighted portfolio; $w_{\text{erc}}$: equal-risk-contribution portfolio; $w_{\text{ref}}$: risk-efficient portfolio; $w_{\text{md}}$: maximum diversification portfolio; $w_{\text{min}}$: minimum volatility portfolio.
Return–sensitive portfolios for $\gamma = 2.7063$ ("medium risk")

<table>
<thead>
<tr>
<th></th>
<th>ir-$w_{mkt}$</th>
<th>ir-$w_{ew}$</th>
<th>ir-$w_{erc}$</th>
<th>ir-$w_{ref}$</th>
<th>ir-$w_{md}$</th>
<th>sample</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Mean</strong></td>
<td>0.027</td>
<td>0.043*</td>
<td>0.045*</td>
<td>0.044</td>
<td>0.041</td>
<td>0.009</td>
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<tr>
<td><strong>Vol.</strong></td>
<td>0.239</td>
<td>0.240</td>
<td>0.235</td>
<td>0.185</td>
<td>0.221</td>
<td>0.388*</td>
</tr>
<tr>
<td><strong>Sharpe</strong></td>
<td>0.114</td>
<td>0.181*</td>
<td>0.190**</td>
<td>0.240</td>
<td>0.184</td>
<td>0.023</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>ir-$w_{mkt}$</th>
<th>ir-$w_{ew}$</th>
<th>ir-$w_{erc}$</th>
<th>ir-$w_{ref}$</th>
<th>ir-$w_{md}$</th>
<th>sample</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Mean</strong></td>
<td>0.024</td>
<td>0.044**</td>
<td>0.044**</td>
<td>0.057</td>
<td>0.030</td>
<td>−0.013</td>
</tr>
<tr>
<td><strong>Vol.</strong></td>
<td>0.262</td>
<td>0.264</td>
<td>0.258</td>
<td>0.199</td>
<td>0.252</td>
<td>0.432**</td>
</tr>
<tr>
<td><strong>Sharpe</strong></td>
<td>0.093</td>
<td>0.165**</td>
<td>0.170**</td>
<td>0.288</td>
<td>0.120</td>
<td>−0.030</td>
</tr>
</tbody>
</table>

Table: Annualized figures. Long only ($c = 1$) and 130%-30% gross constraints ($c = 1.6$) portfolios. ***, ** and * indicate significant differences between the portfolio considered and the market capitalization implied expected return prediction portfolio at the 1%, 5% and 10% level, respectively. Sample denotes results for the naive past return estimation while ir- denotes implied returns obtained for the various proxy portfolios. $w_{mkt}$: market capitalization weighted portfolio; $w_{ew}$: equally-weighted portfolio; $w_{erc}$: equal-risk-contribution portfolio; $w_{ref}$: risk-efficient portfolio; $w_{md}$: maximum diversification portfolio; $w_{min}$: minimum volatility portfolio.
Motivations Outline

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Current research

Has been done:
- Evolution of the slope parameter
- Cross-sectional distribution of implied expected returns over time
- Four-factor regression analysis

To do:
- Sub-window performance analysis
- Expected return performance vs. shrinkage effects
- Alternative covariance matrix estimators
- Dynamically switching between methods
- \texttt{R} package
Thanks for your attention
References


