Nonparametric Estimation of Change Points and Stationarity in Finance

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Change Point Analysis

The process of detecting distributional changes within time ordered data

- Changes is mean, variance, skew, tail, correlation,...

Framework:

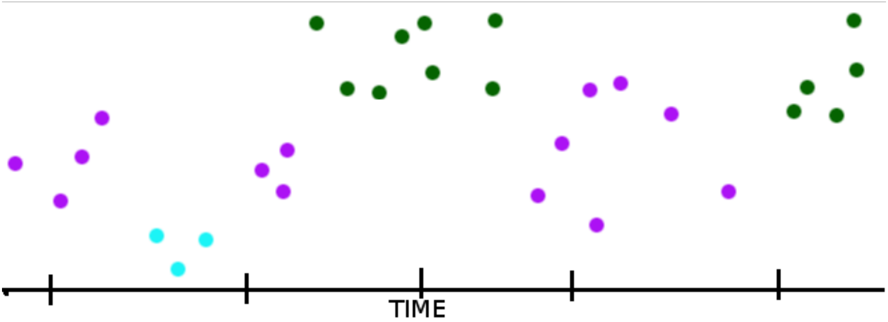
- Retrospective, offline analysis
  - Dynamic Programming for adaptive, online analysis
- Multivariate observations
- Estimation: number of change points and their positions
- Hierarchical algorithms
  - **Divisive:** Data are divided
  - **Agglomerative:** Data are merged

Cluster Analysis

Change point analysis is similar to cluster analysis

In cluster analysis we also wish to partition the observations into homogeneous subsets

- Subsets may not be contiguous in time without some constraints



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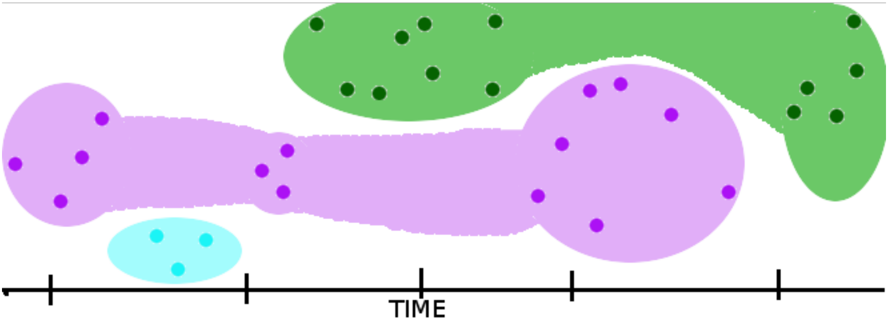

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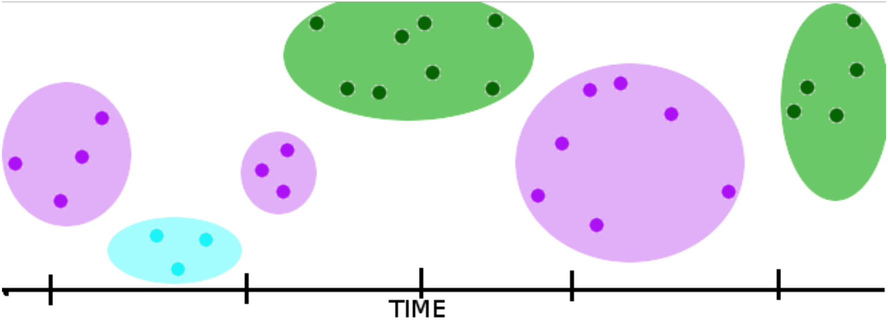
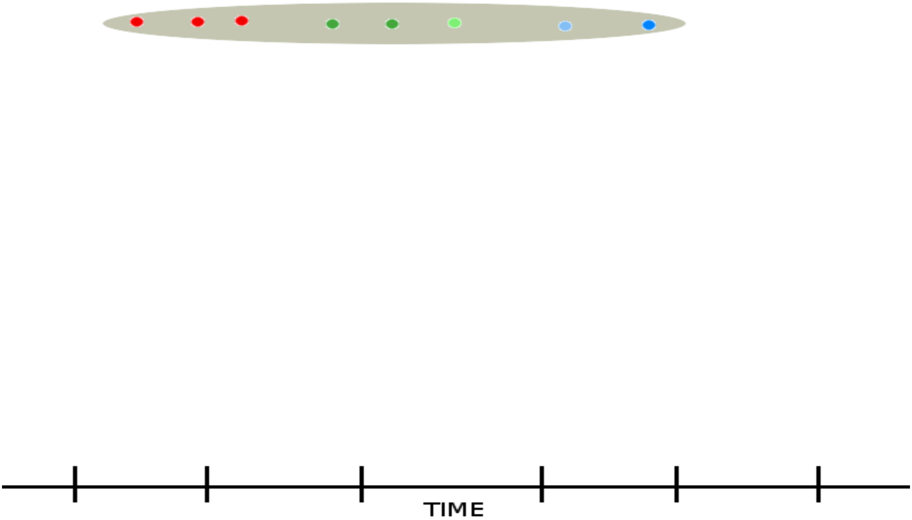


Hierarchical Estimation: Divisive Progression



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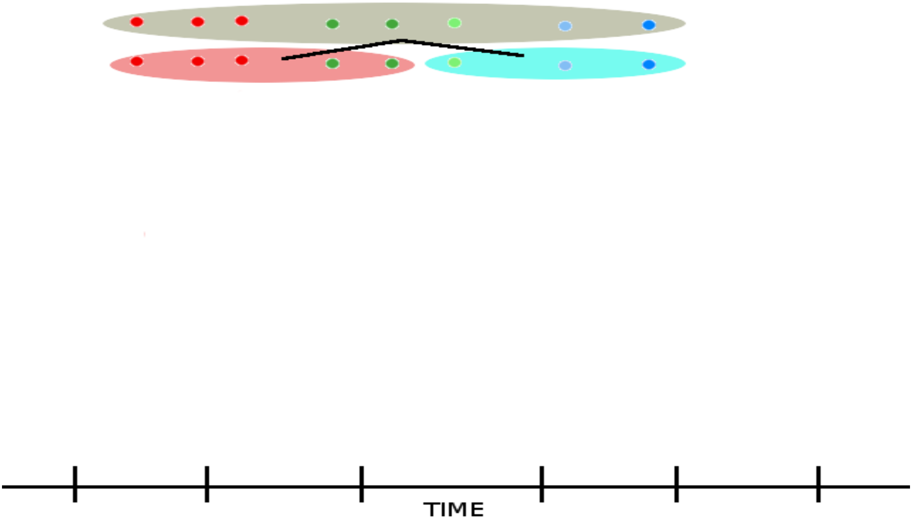

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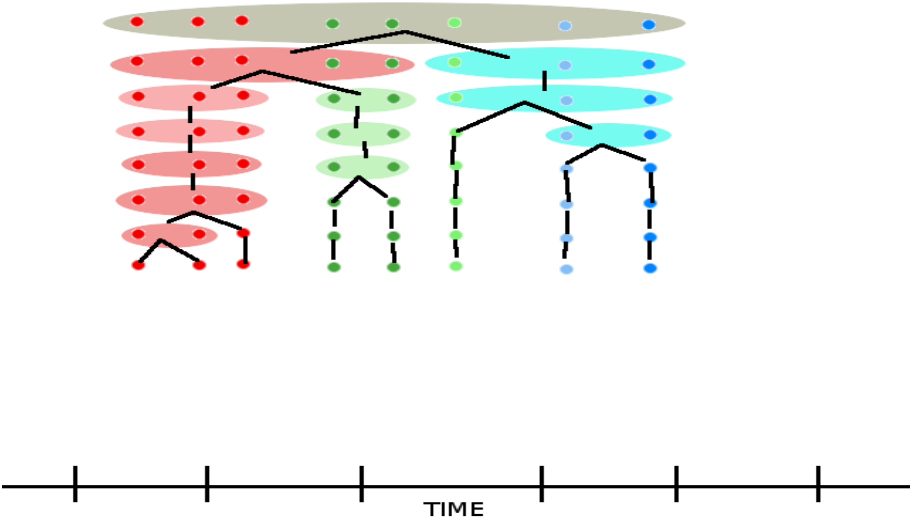

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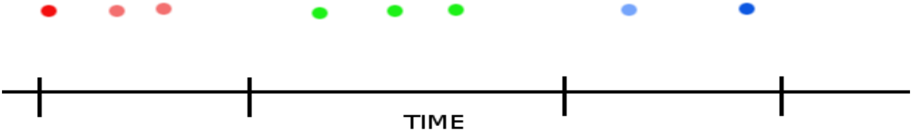

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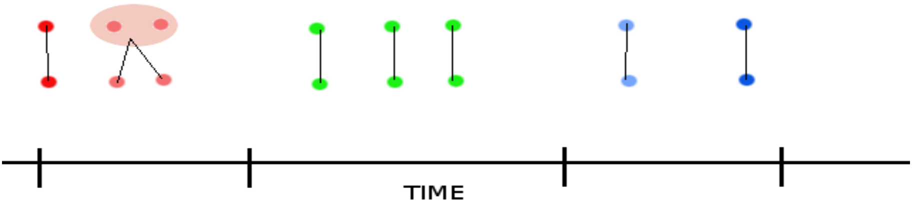

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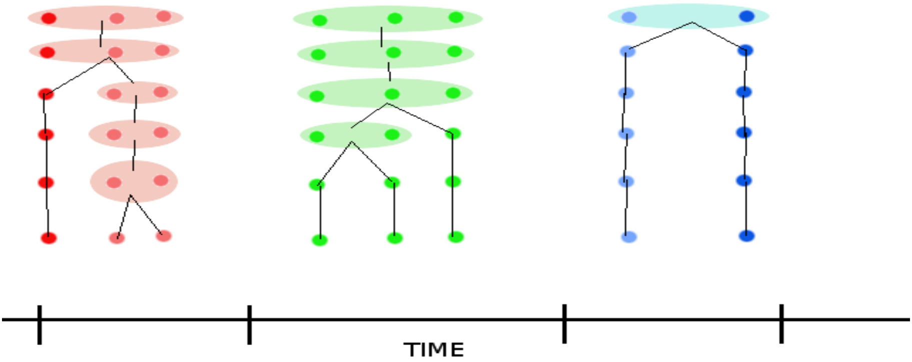

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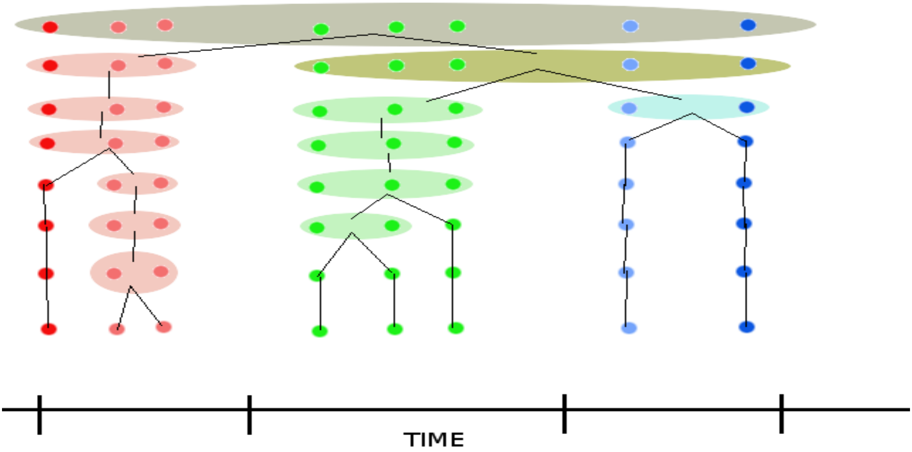
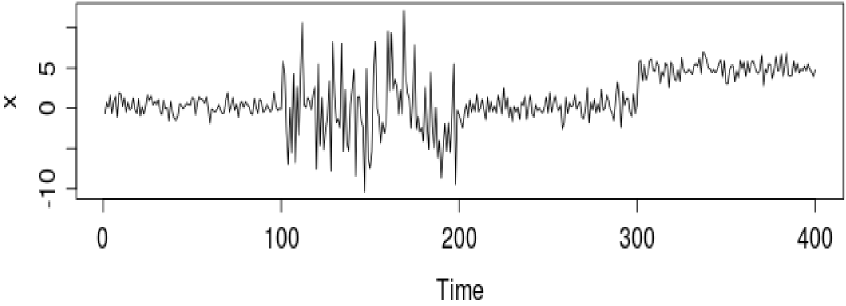


Change Point Analysis

Given independent, time ordered observations $X_1, X_2, \ldots, X_n \in \mathbb{R}^d$

Partition into k homogeneous, temporally contiguous subsets

- k is unknown
- Size of each subset is unknown



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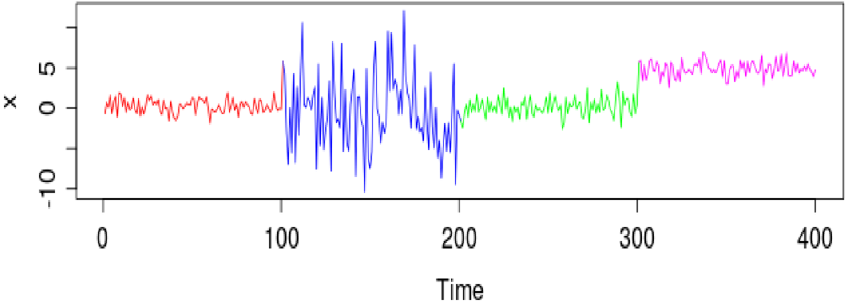

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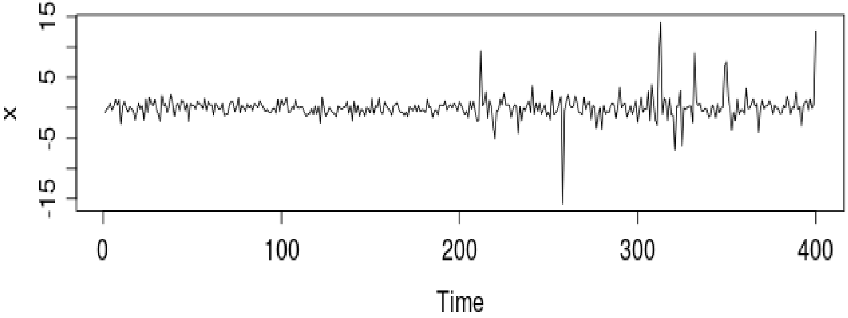

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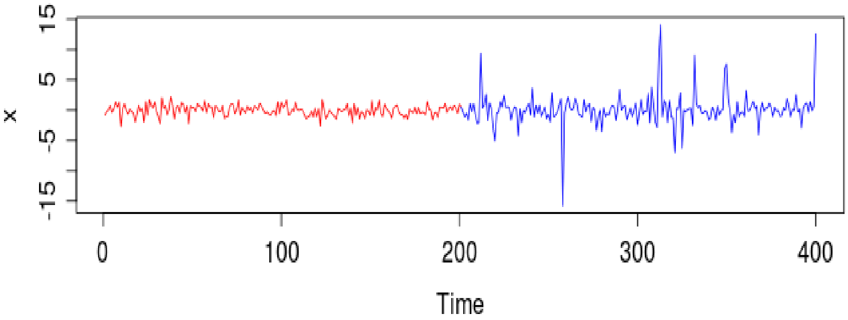
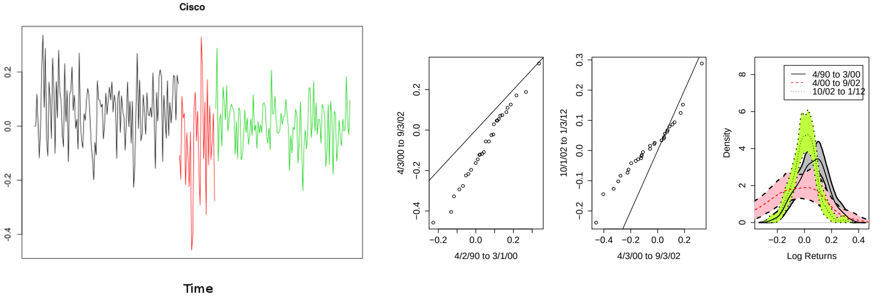


Cisco Systems Inc.

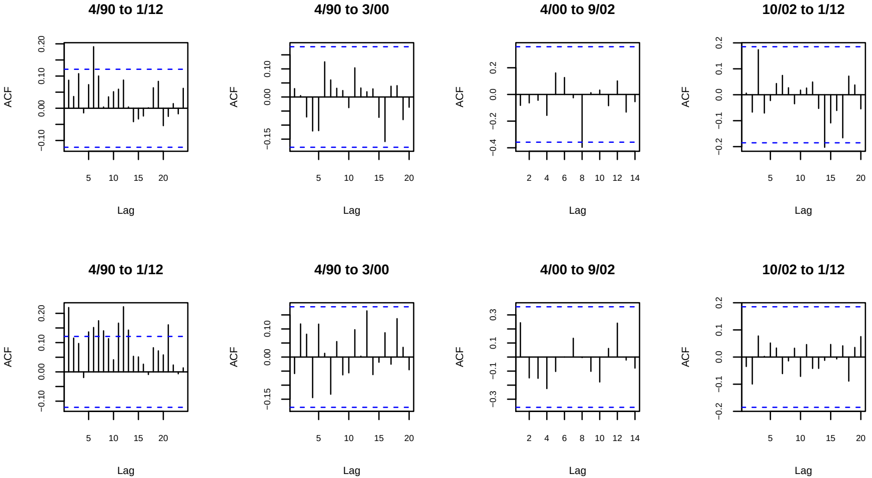
Monthly log returns from April 1990 to January 2010



Change points estimated at April 2000 and October 2002

Cisco Systems

Within cluster diagnostics: $\widehat{ACF}(r_{i,t})$ (top) and $\widehat{ACF}(r_{i,t}^2)$ (bottom)



Measuring Multivariate Homogeneity

Suppose $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^d$ with $\mathbf{X} \sim F_x \perp\!\!\!\perp \mathbf{Y} \sim F_y$

Let $\phi_x(t) = \mathbb{E}\big(e^{i\langle t, \mathbf{X}\rangle}\big)$ and $\phi_y(t) = \mathbb{E}\big(e^{i\langle t, \mathbf{Y}\rangle}\big)$ characteristic functions

Define a divergence between F_x and F_y as

$$\mathcal{E}(\mathbf{X}, \mathbf{Y}; w) = \int_{\mathbb{R}^d} |\phi_x(t) - \phi_y(t)|^2 \, w(t) \, dt,$$

$w(t)$ denotes an arbitrary positive weight function, for which $\mathcal{E}$ exists

Equivalent Divergence Measures

Let $\mathbf{X}$ and $\mathbf{Y}$ be independent, and $(\mathbf{X}', \mathbf{Y}')$ be an iid copy of $(\mathbf{X}, \mathbf{Y})$

Theorem

Suppose that $\mathbb{E}(|\mathbf{X}|^\alpha + |\mathbf{Y}|^\alpha) < \infty$, *for some* $\alpha \in (0, 2]$, *then*

$$\begin{aligned}
\mathcal{E}(\mathbf{X}, \mathbf{Y}; \alpha) &= \int_{\mathbb{R}^d} |\phi_x(t) - \phi_y(t)|^2 \left(\frac{2\pi^{d/2}\Gamma(1-\alpha/2)}{\alpha 2^\alpha \Gamma((d+\alpha)/2)} |t|^{d+\alpha} \right)^{-1} dt \\
&= 2\mathbb{E}|\mathbf{X} - \mathbf{Y}|^\alpha - \mathbb{E}|\mathbf{X} - \mathbf{X}'|^\alpha - \mathbb{E}|\mathbf{Y} - \mathbf{Y}'|^\alpha \\
&< \infty
\end{aligned}$$

- *If* $0 < \alpha < 2$ *then* $\mathcal{E}(\mathbf{X}, \mathbf{Y}; \alpha) = 0$ *if and only if* $\mathbf{X}$ *and* $\mathbf{Y}$ *are identically distributed*
- *If* $\alpha = 2$ *then* $\mathcal{E}(\mathbf{X}, \mathbf{Y}; \alpha) = 0$ *if and only if* $\mathbb{E}\mathbf{X} = \mathbb{E}\mathbf{Y}$ *(equal means)*

An Empirical Measure (U-statistics)

Let $\mathbf{X}_n = \{X_i : i = 1, \ldots, n\}$ and $\mathbf{Y}_m = \{Y_j : j = 1, \ldots, m\}$ be independent iid samples from the distribution of $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^d$, respectively, such that $E|\mathbf{X}|^\alpha, E|\mathbf{Y}|^\alpha < \infty$ for some $\alpha \in (0, 2]$

Define

$$\widehat{\mathcal{E}}(\mathbf{X}_n, \mathbf{Y}_m; \alpha) =$$

$$\frac{2}{mn} \sum_{i=1}^{n} \sum_{j=1}^{m} |X_i - Y_j|^\alpha - \binom{n}{2}^{-1} \sum_{1 \le i < k \le n} |X_i - X_k|^\alpha - \binom{m}{2}^{-1} \sum_{1 \le j < k \le m} |Y_j - Y_k|^\alpha$$

and

$$\widehat{\mathcal{Q}}(\mathbf{X}_n, \mathbf{Y}_m; \alpha) = \frac{mn}{m+n} \widehat{\mathcal{E}}(\mathbf{X}_n, \mathbf{Y}_m; \alpha)$$

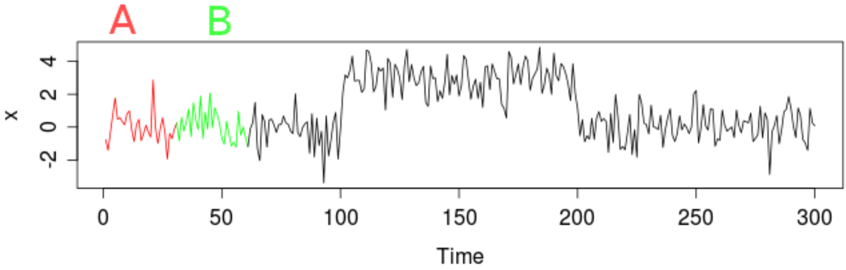
The E-Divisive Algorithm: Estimating Location

$\mathbf{A}_\tau = \{Z_1, Z_2, \ldots, Z_\tau\}$ and $\mathbf{B}_\tau(\kappa) = \{Z_{\tau+1}, Z_{\tau+2}, \ldots, Z_\kappa\}$

A change point location $\hat{\tau}$ is estimated as

$$(\hat{\tau}, \hat{\kappa}) = \underset{(\tau,\kappa)}{\operatorname{argmax}} \; \widehat{\mathcal{Q}}(\mathbf{A}_\tau, \mathbf{B}_\tau(\kappa); \alpha)$$

Thus, we maximize $\frac{mn}{n+m}\widehat{\mathcal{E}}(\mathbf{A}, \mathbf{B}; \alpha)$ for all subsets **A** and **B**:



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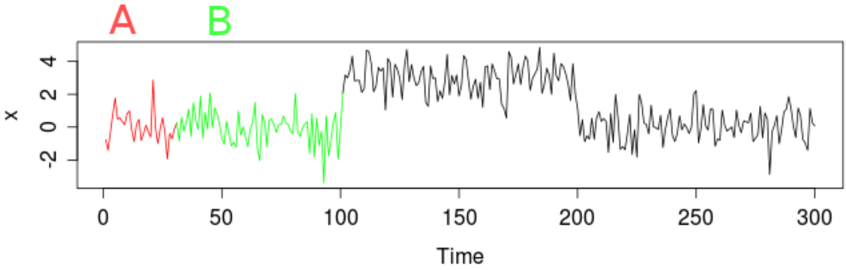

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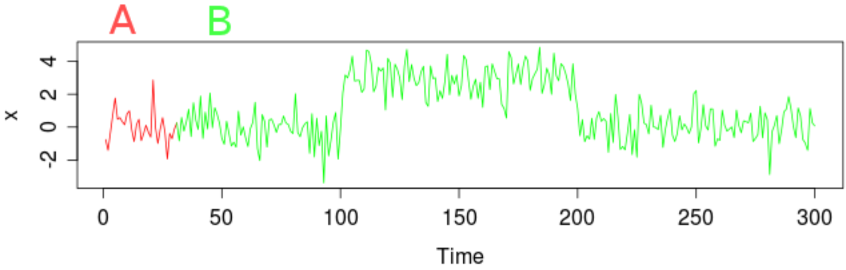

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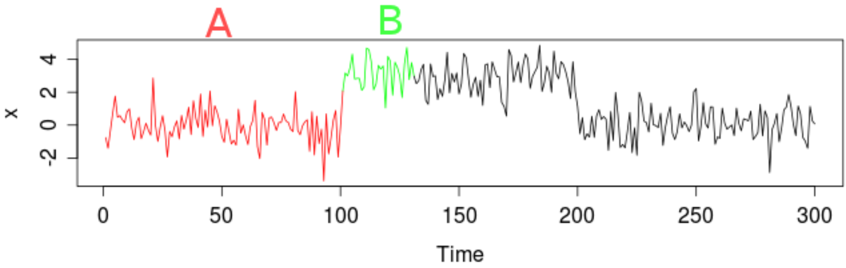

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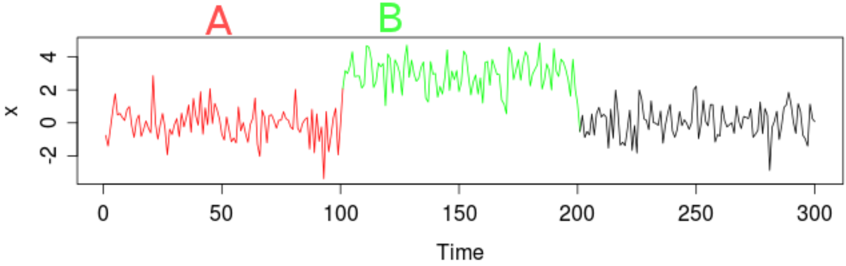

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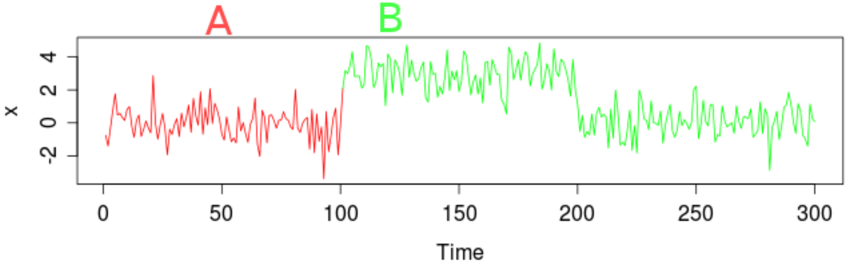

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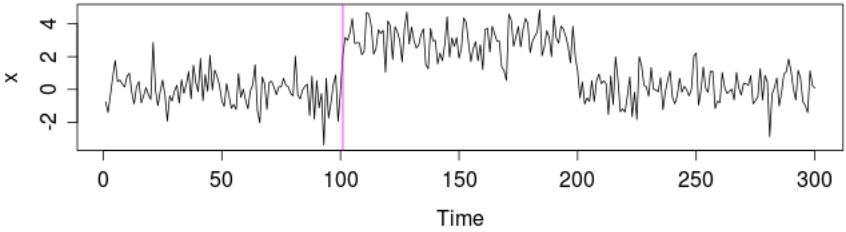

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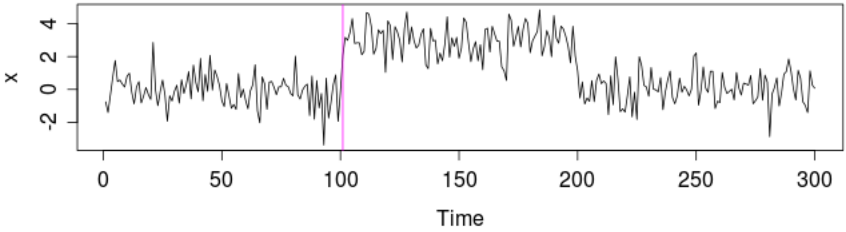
Distribution of test statistic $\hat{q}^* = \widehat{\mathcal{Q}}(\mathbf{A}_\tau, \mathbf{B}_\tau(\kappa); \alpha)\big|_{\tau=\hat{\tau}}$ is unknown

Significance of proposed change point measured via permutation test

Randomly permute series, maximize $\frac{mn}{n+m}\widehat{\mathcal{E}}(\mathbf{A}, \mathbf{B}; \alpha)$, record and repeat:



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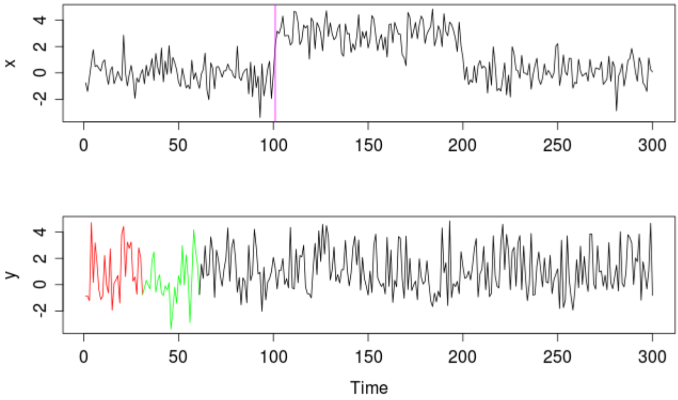

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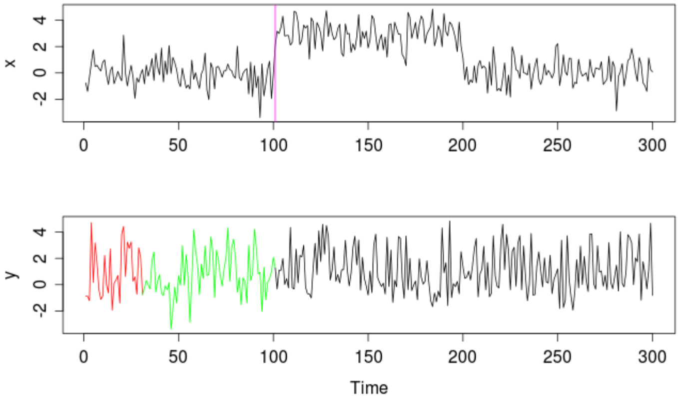

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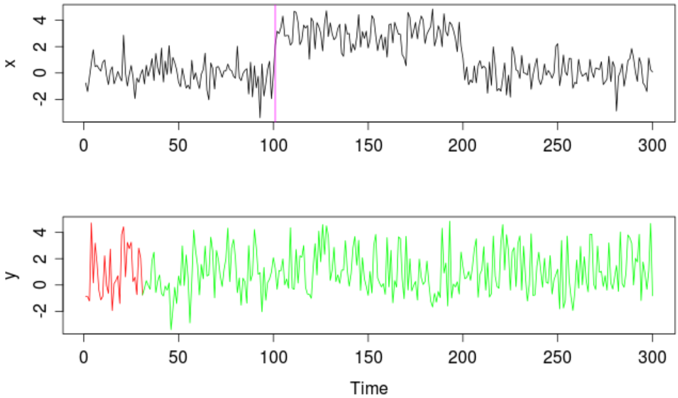

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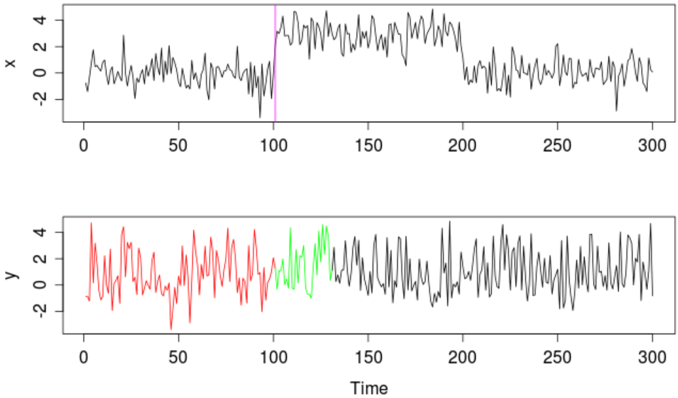

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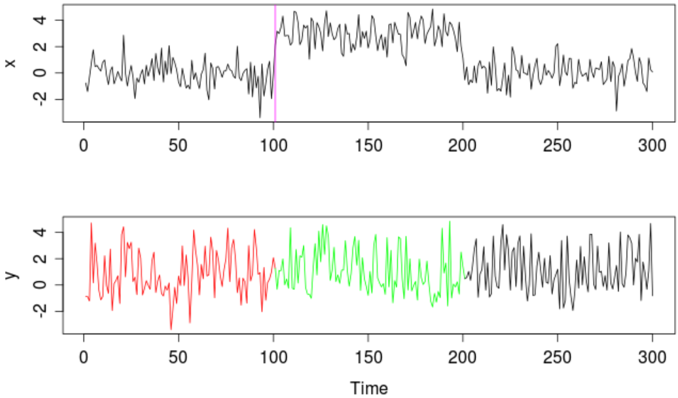

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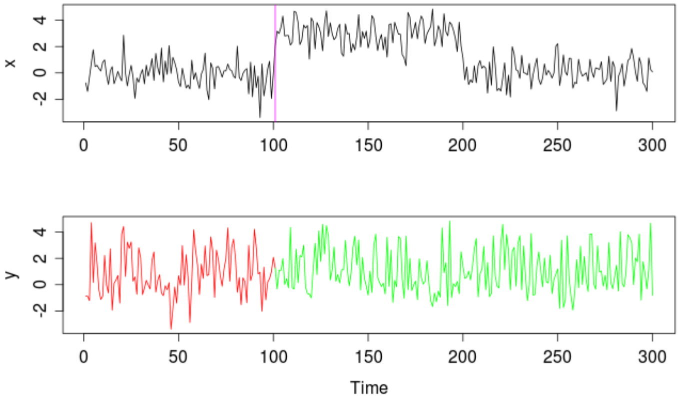

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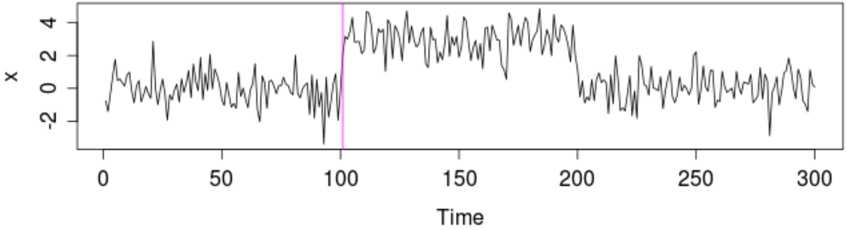

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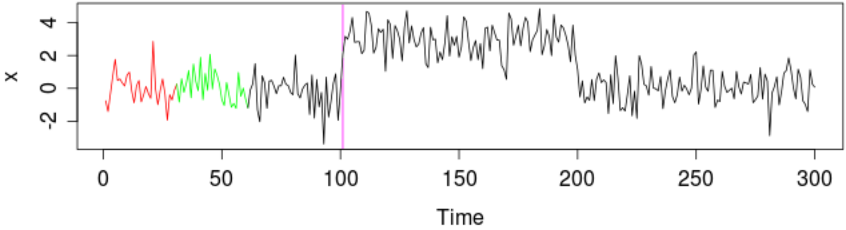

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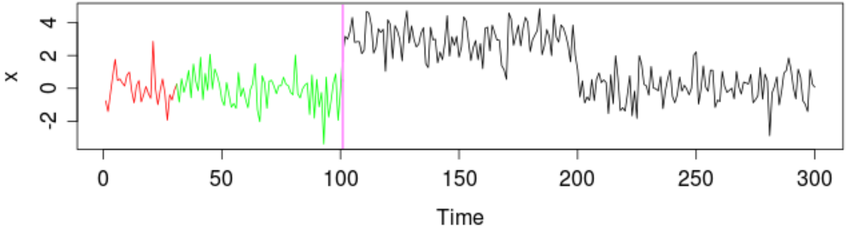

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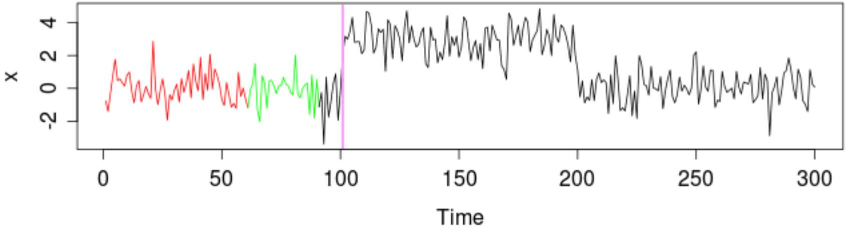

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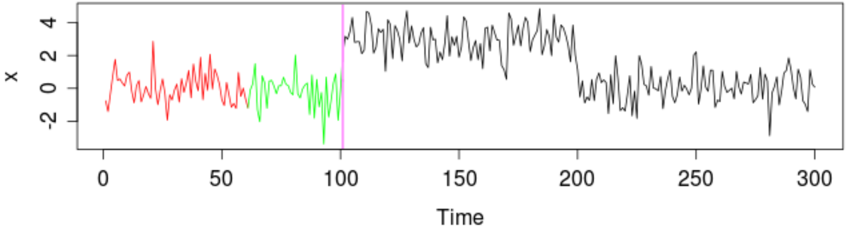

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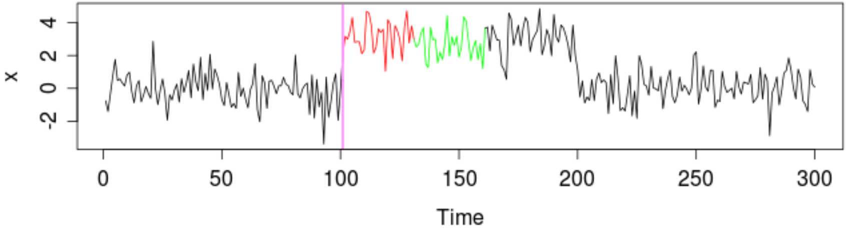

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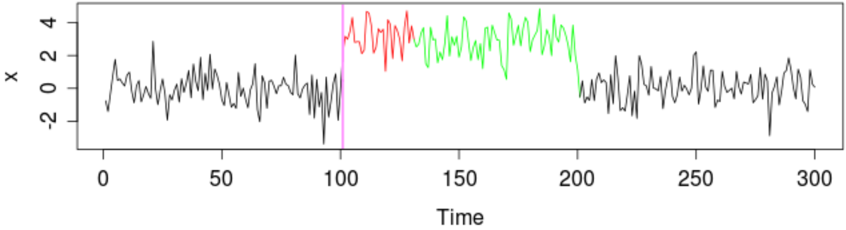

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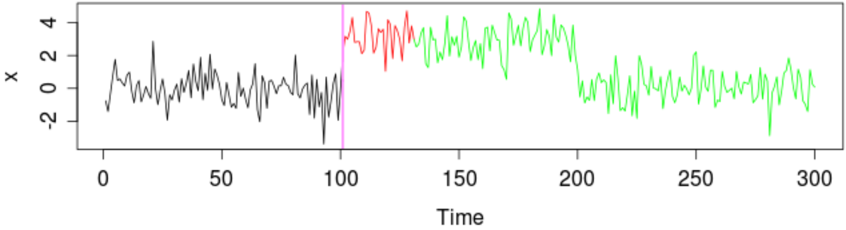
If $\hat{q}^* = \widehat{\mathcal{Q}}(\mathbf{A}_\tau, \mathbf{B}_\tau(\kappa); \alpha)\Big|_{\tau=\hat{\tau}}$ is insignificant: STOP

If significant, condition on location, and repeat within clusters:



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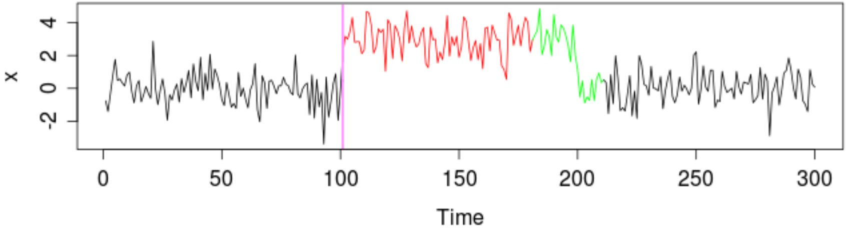

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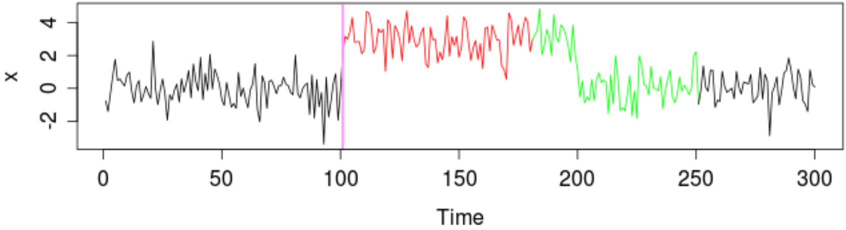

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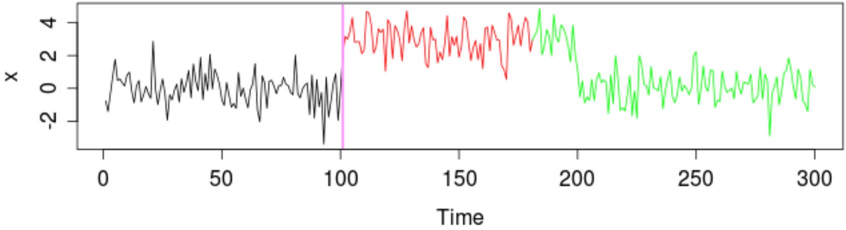

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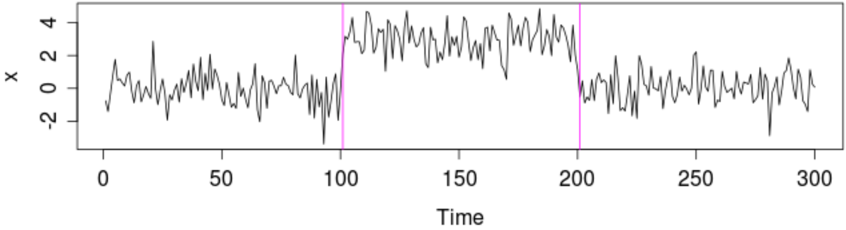
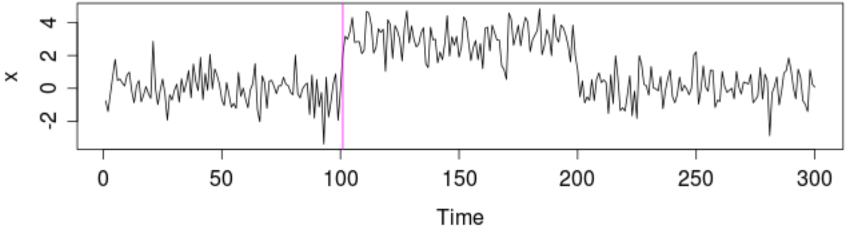


The E-Divisive Algorithm: Multiple Change Points

Once again, perform permutation test

However, only permute within each cluster:



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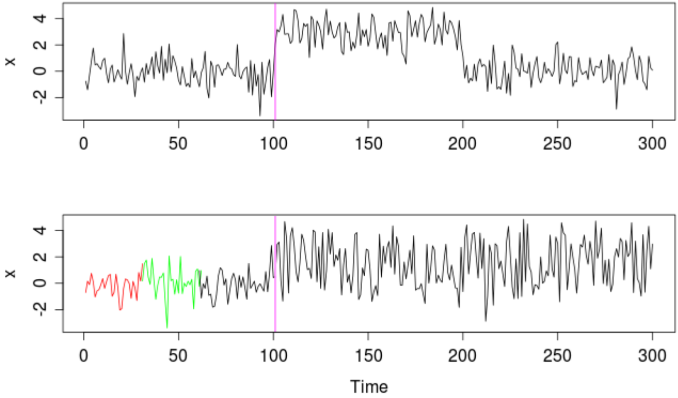

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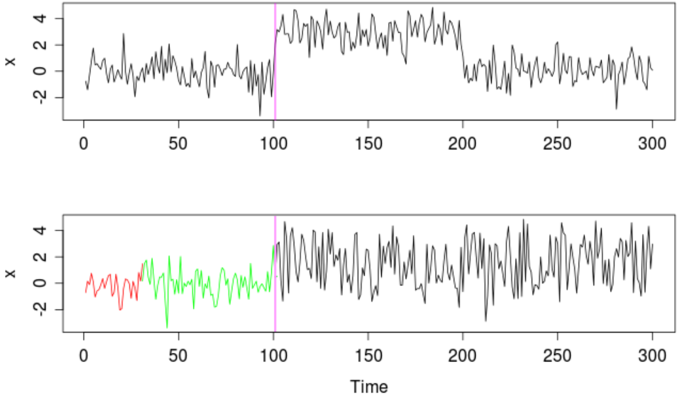

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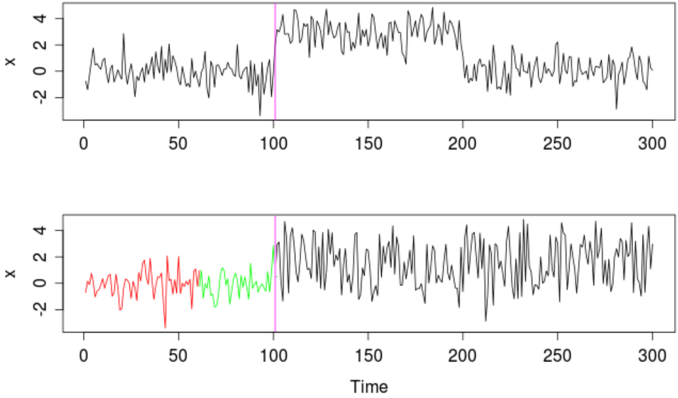

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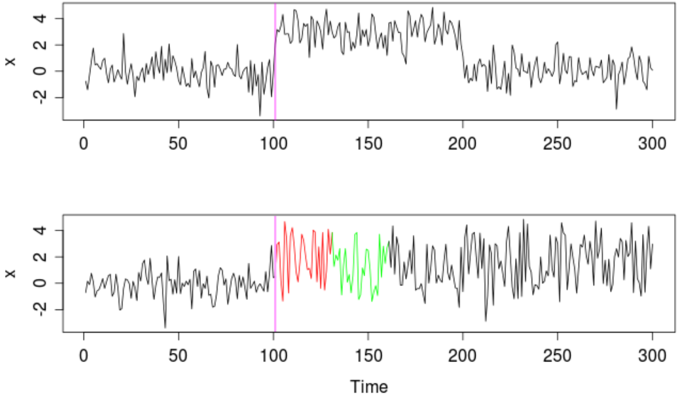

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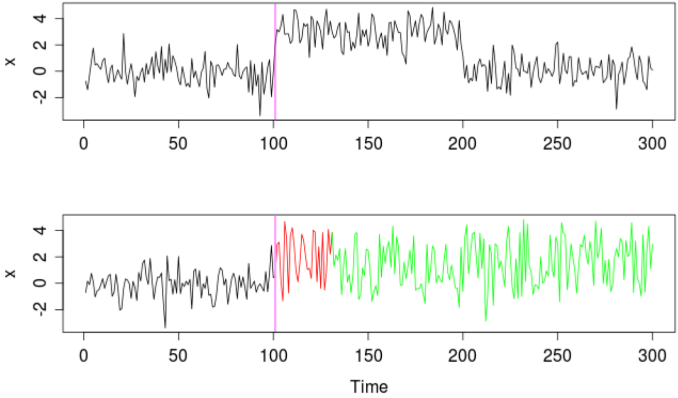

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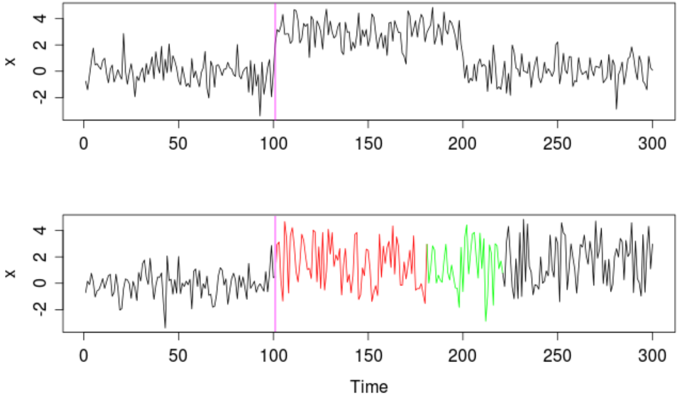

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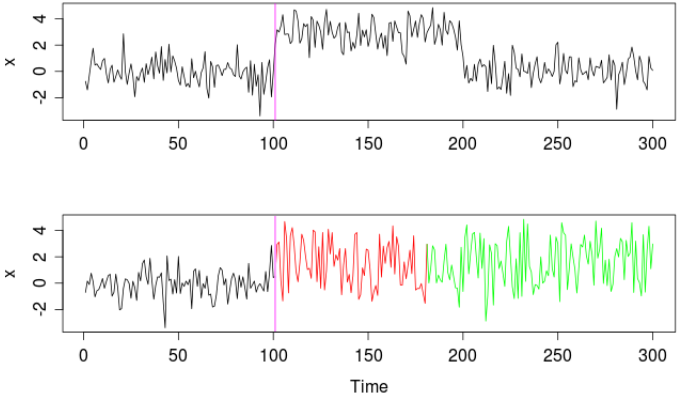


The 'ecp' R package (CRAN)

Signature:

```
e.divisive(X, sig.lvl=0.05, R=199, k=NULL, min.size=30, alpha=1)
```

Arguments:

- `X` - A $T \times d$ matrix representation of a length T time series, with d-dimensional observations.
- `sig.lvl` - The significance level used for the permutation test.
- `R` - The maximum number of permutations to perform in the permutation test.
- `k` - The number of change points to return. If this is NULL only the statistically significant estimated change points are returned.
- `min.size` - The minimum number of observations btw change points.
- `alpha` - The index for test statistic.

The ‘ecp’ R package (CRAN)

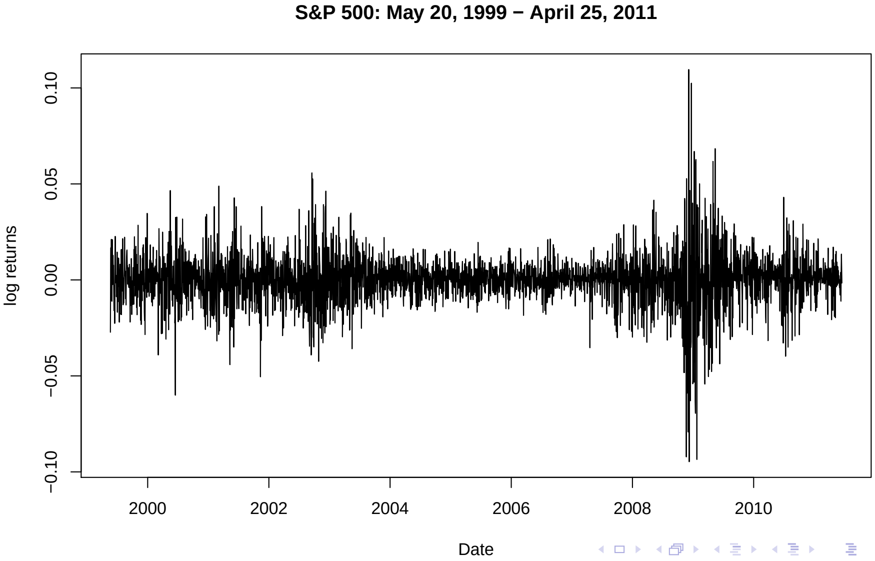
Returned list:

- `estimates` - The vector containing the estimated change point locations.
- `cluster.number` - The number of segments created by the estimated change points.
- `considered.last` - The location of the last estimated change point that was not deemed statistically significant.
- `order.found` - The order in which the change points were estimated.
- `Pvalues` - The approximate p-values returned by the permutation test.

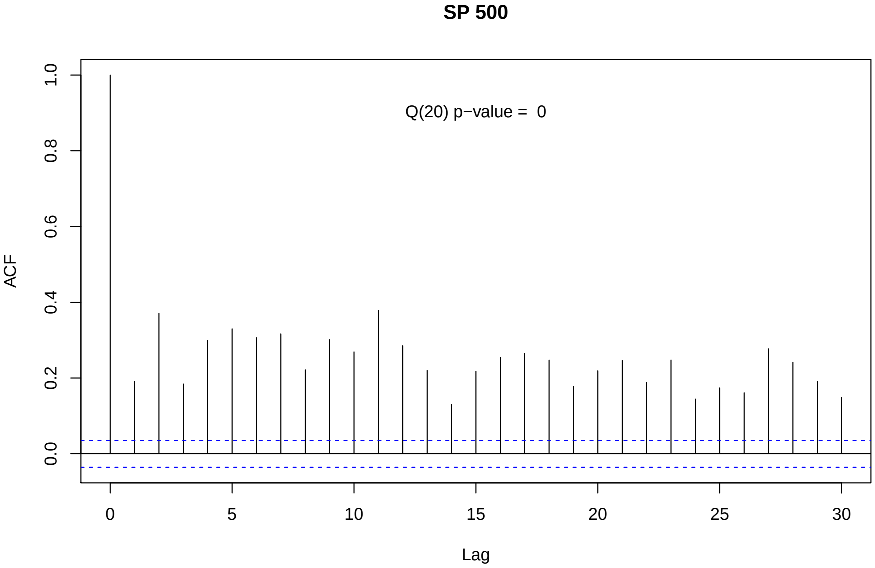
Complexity is $\mathcal{O}(kT^2)$

S&P 500 Index

The E-divisive procedure was applied to daily S&P 500 Index log returns

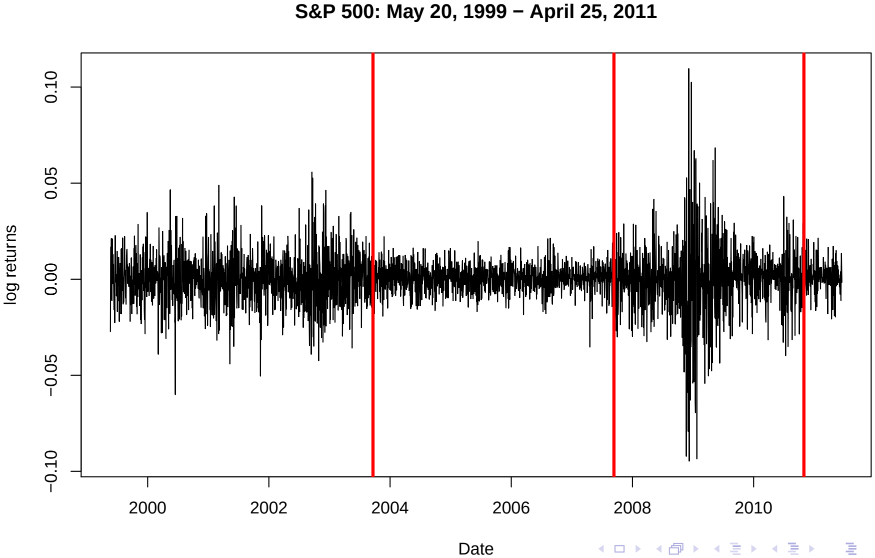


S&P 500 Index: Very significant ARCH effect overall



S&P 500 Index: Regime-Switching GARCH model

The E-divisive procedure was applied to daily S&P 500 Index log returns



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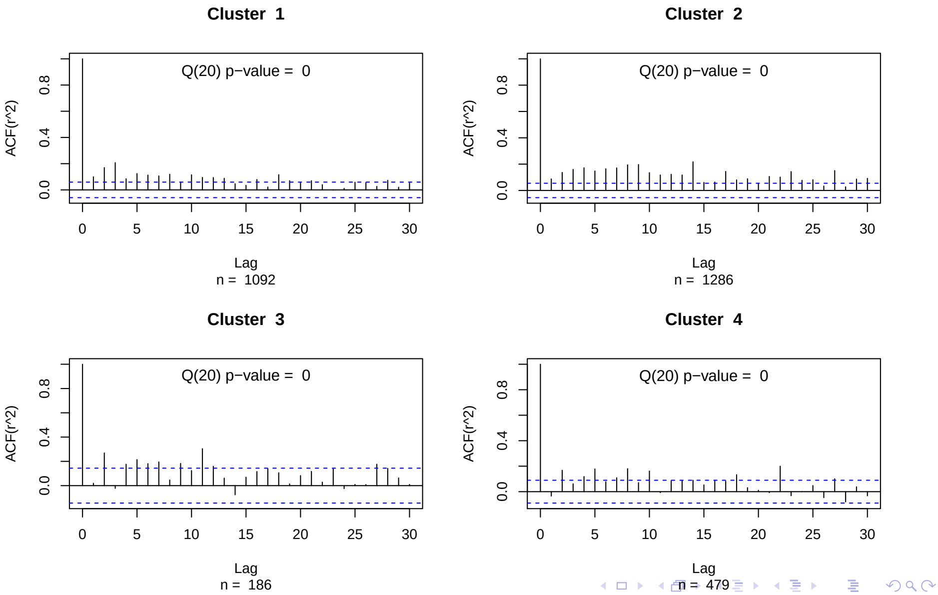
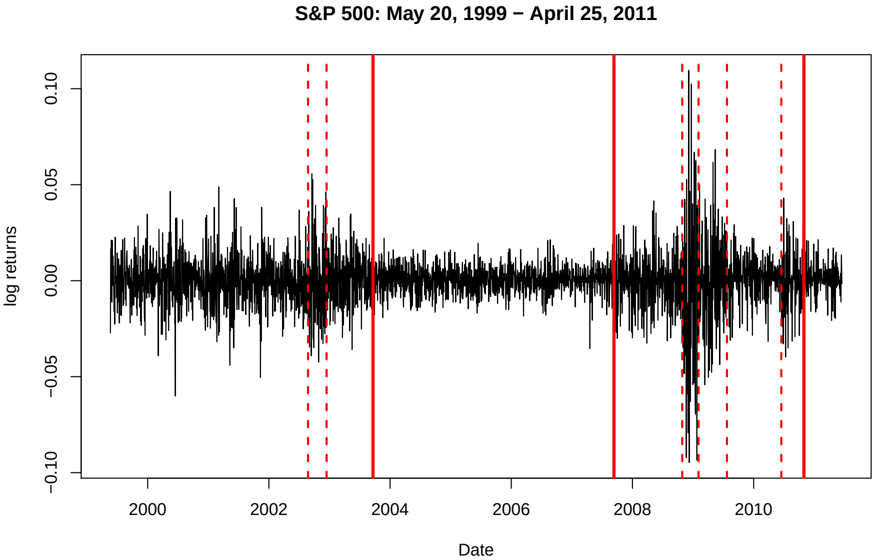
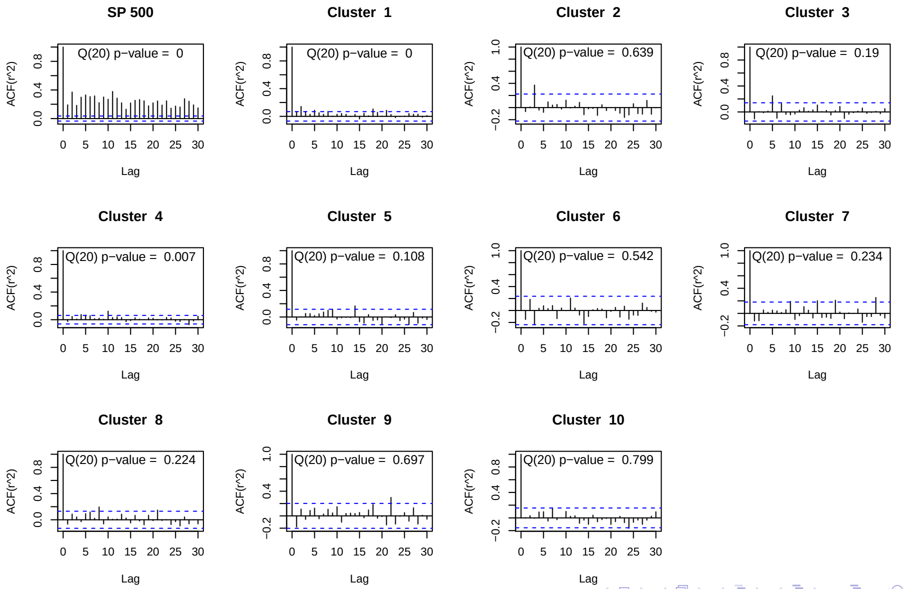


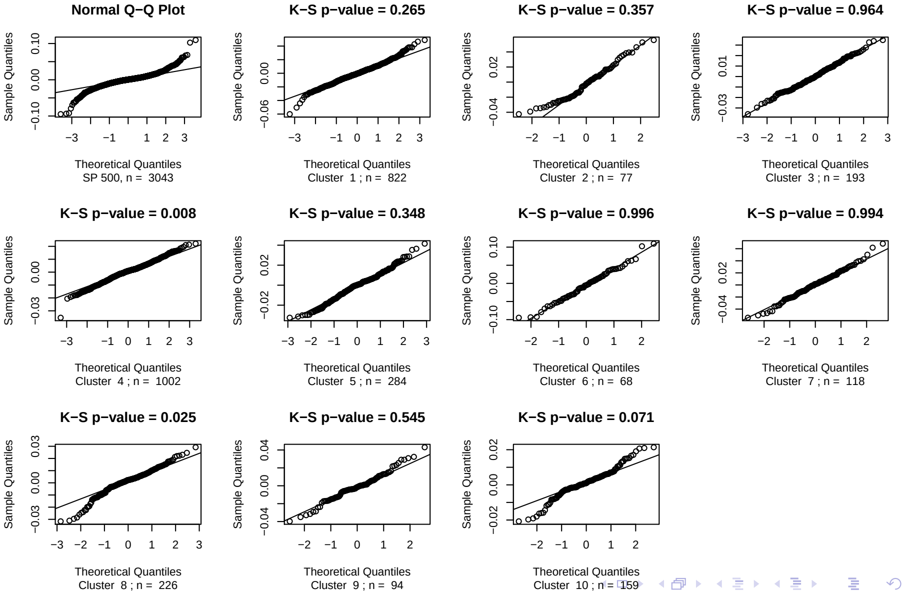
S&P 500 Index: Apply E-divisive Algorithm



S&P 500 Index: No ARCH effect within clusters!



S&P 500 Index: Gaussian distribution within clusters?



Multivariate Change in Correlation

1,000 simulations, 2 CP: $N_d(0,\Sigma)$, $N_d(0,I)$, $N_d(0,\Sigma)$

$$\Sigma_{w.o./noise} = \begin{pmatrix} 1 & \rho & \rho & \cdots & \rho \\ \rho & 1 & \rho & \cdots & \rho \\ \rho & \rho & 1 & \cdots & \rho \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho & \rho & \rho & \cdots & 1 \end{pmatrix} \qquad \Sigma_{w/noise} = \begin{pmatrix} 1 & \rho & 0 & \cdots & 0 \\ \rho & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

E-Divisive		Average Rand Index	
T	d	**Without Noise**	**With Noise**
300	2	0.767	0.774
	5	0.912	0.736
	9	0.970	0.736
600	2	0.817	0.836
	5	0.993	0.631
	9	0.998	0.666
900	2	0.970	0.968
	5	0.998	0.644
	9	0.999	0.612

Local Stationarity

A d-dimensional process $\{Y_t\}_{t=1}^{T}$ is strictly stationary if

$$F_{y_1,\ldots,y_k}(Y_1,\ldots,Y_k) = F_{y_{1+\tau},\ldots,y_{k+\tau}}(Y_{1+\tau},\ldots,Y_{k+\tau})$$

$\forall k, \tau \in \mathbb{N}$, in which F denotes a joint distribution function

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An equivalent condition is that $\forall k, \tau \in \mathbb{N}$

$$\phi_{y_1,\ldots,y_k}(s) = \phi_{y_{1+\tau},\ldots,y_{k+\tau}}(s),\ \forall s \in \mathbb{R}^{d\times k}$$

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Piecewise stationary:

$\forall t$, $\exists w_t \geq 0$, such that all observations in $[t - w_t, t]$ are stationarity process

$\underline{w_t}$ defines a one-sided window of homogeneity at t

Cointegration

Bivariate process $X_t = (x_{1,t}, x_{2,t})'$ is cointegrated if

1. Each component is I(1) (unit root nonstationary)
2. $\exists \beta \neq 0$ such that $\underline{x_{1,t} - \beta x_{2,t}}$ is I(0) (unit root stationary)

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Error Correction Model (ECM):

For bivariate I(1) process X_t, with cointegrating vector $\boldsymbol{\beta} = (1, -\beta)'$ one-lag ECM

$$\Delta X_t = \boldsymbol{\mu} + \boldsymbol{\alpha}\boldsymbol{\beta}' X_{t-1} + \boldsymbol{\Phi}\Delta X_{t-1} + \varepsilon_t$$

$\Delta X_t = X_t - X_{t-1}$; $\varepsilon_t \overset{\text{iid}}{\sim} (\mathbf{0}, \boldsymbol{\Sigma})$; and $\boldsymbol{\mu}, \boldsymbol{\alpha}, \boldsymbol{\Phi}, \boldsymbol{\Sigma}$ are constant matrices

Local Cointegration

A bivariate process X_t is locally cointegrated with respect to a window of homogeneity w_t if $\forall t$

1. X_t is I(1), within the interval $[t - w_t, t]$ (unit root nonstationary)
2. $\exists \beta_t \neq 0$ such that $\underline{u_t = x_{1,t} - \beta_t x_{2,t}}$ is I(0) (unit root stationary)

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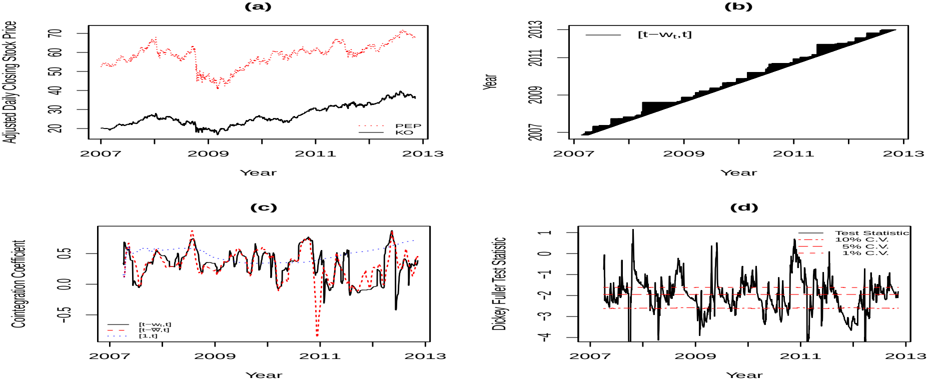
At time t, find largest δ such that $(Z_t, u_t)'$ is stationary over $[t - \delta + 1, t]$

- $Z_t = \Delta X_t$
- $u_t = x_{1t} - \beta x_{2t}$
- β estimated via OLS over $[t - \delta + 1, t]$

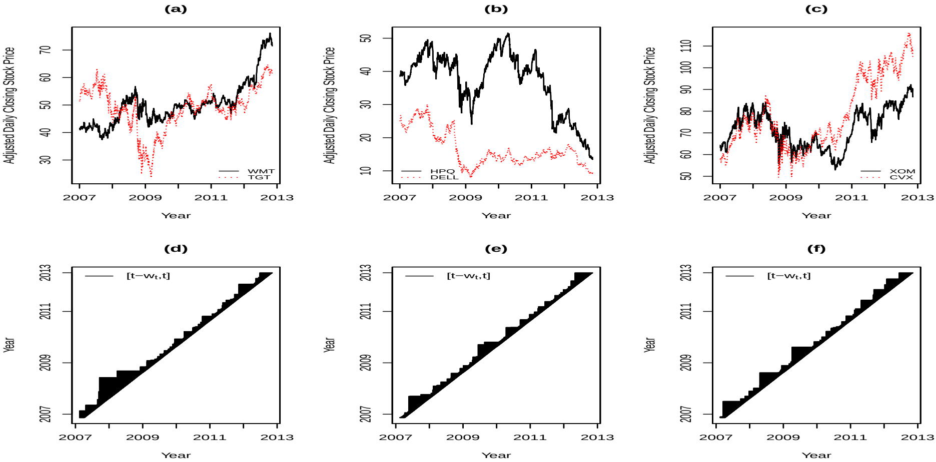
(a) Pepsi (PEP) and Coca-Cola (KO) adjusted daily closing stock prices January 2007 through November 2012

(b) $[t - w_t, t]$, estimated window of local stationarity at times t



(c) Cointegrating coefficient $\hat{\beta}(t, w_t)$ over $[t - w_t, t]$ (adaptive); $\hat{\beta}(t, \bar{w} = 68)$ over $[t - 68, t]$ (fixed); and $\hat{\beta}_t$, over $[1, t]$ (cumulative)

(d) Dickey-Fuller test statistic over $[t - w_t, t]$ (1, 5, & 10 % critical values) (abc) Adjusted daily closing prices from 1/2007 through 11/2012 of Walmart (WMT) & Target (TGT), Hewlett Packard (HPQ) & Dell (DELL), and Exxon Mobil (XOM) & Chevron (CVX)



(def) Estimated window of local stationarity $[t - w_t, t]$, at times t

Pairs Trading: Trading the Spread $\hat{u}_t = x_{1,t} - \hat{\beta}_t(\hat{w}_t) x_{2,t}$

ENTER at $|2\hat{\sigma}_t|$

EXIT at $|\frac{1}{2}\hat{\sigma}_t|$ (gain), $|3\hat{\sigma}_t|$ (loss), or $T = t + \delta_{min}$ (time limit)

Mean Return (per trade)

Window/Pair	KO-PEP	HPQ-DELL	WMT-TGT	XOM-CVX
Fixed	-7.0%	2.5%	-6.6%	0.9%
Cumulative	12.0%	-0.9%	-0.1%	-1.8%
Adaptive	21.0%	4.4%	1.1%	43.0%

Table: Mean return (per trade) for the three window methods on each pair

Mean Trade Duration (days)

Window/Pair	KO-PEP	HPQ-DELL	WMT-TGT	XOM-CVX
Fixed	7.8	9.7	9.7	8.5
Cumulative	16.9	23.0	13.4	17.8
Adaptive	7.8	8.2	8.3	10.6

Table: Mean trade duration (days) for the three window methods on each pair

Bibliography

http://www.stat.cornell.edu/~matteson/

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