Asset Allocation with Higher Order Moments and Factor Models

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Based on joint research with:

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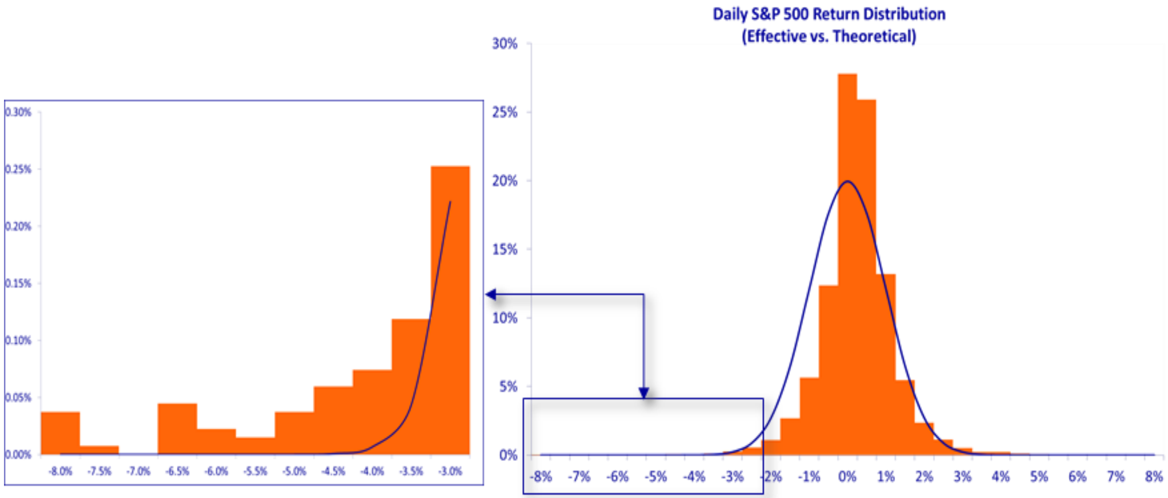
- The world of asset returns is non-normal. Its distribution tends to be asymmetric and extremes occur too often to be compatible with the tails of a normal distribution.



- If there is no estimation error, most investors would be willing to sacrifice expected return and/or accept a higher volatility in exchange for a higher skewness and lower kurtosis leading to a lower downside risk (e.g. Ang, Chen and Xing, 2006; Harvey and Siddique, 2000, Scott and Horvath, 1980).

- This trade-off between positive preferences for odd moments (mean, skewness) and negative preferences for even moments (variance, kurtosis) can be conveniently summarized into a single objective function
  – using a Taylor expansion of the expected utility function as objective (Jondeau and Rockinger, 2006; Martellini and Zieman, 2010)
  – or a portfolio downside risk objective based on the Cornish-Fisher expansion (e.g. Peterson and Boudt, 2008; Boudt et al., 2013).

Estimation and decomposition of downside risk for portfolios with non-normal returns

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Technical paper

Component VAR for a non-normal world

Author: Brian Peterson, Kris Boudt
Source: Risk magazine | 01 Nov 2008
Categories: Market Risk

R package PerformanceAnalytics

Asset allocation with conditional value-at-risk budgets

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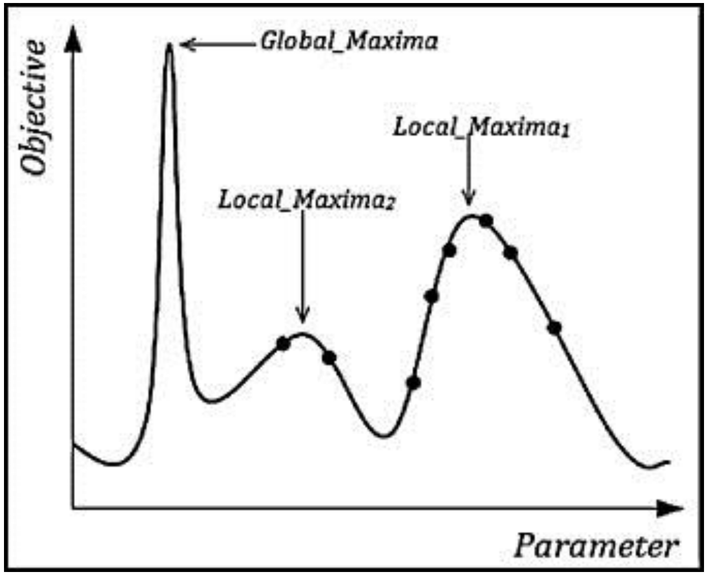
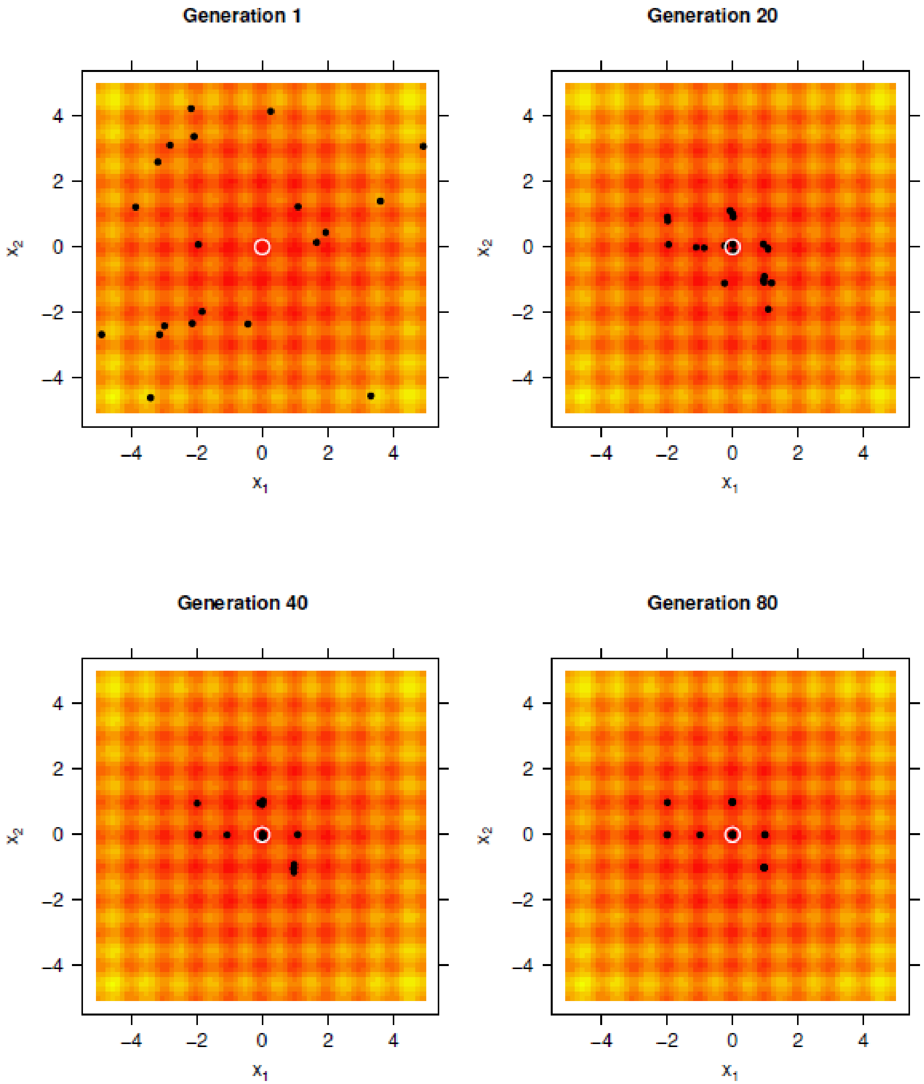
Higher order comoments are also needed in the portfolio constraints; eg. Downside risk budgets

…together with constraints on the portfolio turnover, cardinality constraints, … this makes the portfolio problem complex to solve…



R packages Deoptim and PortfolioAnalytics



Differential Evolution with the R package DEoptim:
Cross-over and mutations gradually improve the initial population and get close to the global optimum.

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Differential Evolution with DEoptim

An Application to Non-Convex Portfolio Optimization

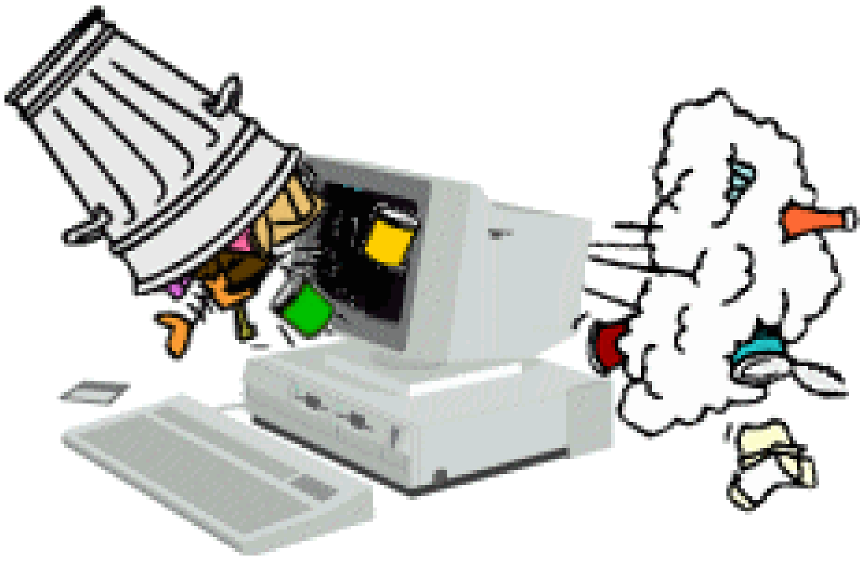
by David Ardia, Kris Boudt, Peter Carl, Katharine M. Mullen and Brian G. Peterson

Abstract The R package **DEoptim** the Differential Evolution algorith gorithm is an evolutionary techniq classic genetic algorithms that is u solution of global optimization prob note we provide an introduction to and demonstrate its utility for fin cations by solving a non-convex p mization problem.

tween upper and lower bounds (defined or using values given by the user. Each involves creation of a new population fr rent population members $\{x_i \mid i = 1, \ldots,$

Smart beta portfolios optimizing risk under various constraints

- This all leads to a powerful portfolio optimization framework, provided that the optimization problem is sensible and the input parameters are of good quality.



- The challenge is that while the sample size in terms of history of returns is often limited, many problems require to optimize over a large number of assets:
  - Curse of dimensionality in the number of parameters ➔ large estimation errors.
- The consequences of estimation error in portfolio optimization are well known:
  - Optimized portfolios are often not well-diversified (Green and Hollifield, 1993)
  - and behave like “error maximizers” (Michaud, 1998).

- The optimization framework has to be *reliable, stable, and robust with respect to model and estimation errors.*
  - Impose constraints (weights and others)
  - Clean the data;
  - Improve the accuracy of the estimates:
    - Resampling
    - Impose structure:
      - Factor model;
      - Possibly combined with shrinkage of the estimators toward a target;
      - Large literature for the mean and the covariance matrix

Contribution

- Only 1 paper for **coskewness and cokurtosis**: Martellini and Zieman (RFS, 2010): equicorrelation and single factor model.
- Single factor assumption *may* make sense for equities but our application is on asset allocation (bonds, equities, commodities)
- We thus extend Martellini and Zieman's paper to the multifactor model.
- We investigate the out-of-sample gains in an asset allocation framework with 17 assets.

Outline

- Why we need higher order comoments for portfolio optimization and curse of dimensionality in using sample based estimators;
- Estimation of higher order comoments under the multifactor model;
- Empirical analysis;
- Conclusion and further research.

THE NEED FOR HIGHER ORDER COMOMENTS AND THE CURSE OF DIMENSIONALITY

- Many investors care about higher order portfolio moments;
- When optimizing the portfolio using genetic algorithms like DE, these moments need to be calculated thousands of time;
- Higher order comoment matrices allow us to express portfolio moments as explicit, rapid to evaluate functions of the portfolio weights:

The first four portfolio moments and how they relate to comoments

$$E[w'R] = w'\mu$$
$$E\left[\left(w'(R-\mu)\right)^2\right] = w'\Sigma w$$
$$E\left[\left(w'(R-\mu)\right)^3\right] = w'\Phi(w \otimes w)$$
$$E\left[\left(w'(R-\mu)\right)^4\right] = w'\Psi(w \otimes w \otimes w).$$

Needed eg to opimize expected utility under the 4th order taylor expansion; to estimate cornish-fisher VaR and ES, etc.

Nx1	$\mu = E[R]$
NxN	$\Sigma = E\left[(R-\mu)(R-\mu)'\right]$
NxN^2	$\Phi = E\left[(R-\mu)(R-\mu)' \otimes (R-\mu)'\right]$
NxN^3	$\Psi = E\left[(R-\mu)(R-\mu)' \otimes (R-\mu)' \otimes (R-\mu)'\right]$

$\otimes$: *Kronecker Product*

Curse of dimensionality: Number of unique elements as a function of N

Covariance matrix Σ:
N(**N**-1)/2 elements

Coskewness matrix Φ:
N(**N**+1)(**N**+2)/6 elements

Cokurtosis matrix Ψ:
N(**N**+1)(**N**+2)(**N**+3)/24 elements

Total number of unique elements:
N(N+1)/2+ N(N+1)(N+2)/6 + **N**(**N**+1)(**N**+2)(**N**+3)/24

	Number of elements to estimate
N=5	120
N=20	10,605
N=100	4,598,025

To be compared with the number of time series observations available in typical portfolio allocations with monthly rebalancing:
5 years: 60 (monthly), 260 (weekly returns):
Large estimation errors in the sample based estimators and risk of “error maximizing” when optimizing

$$\hat{\Sigma} = \frac{1}{T}\sum_{t=1}^{T} \left[(R_t - \hat{\mu})(R_t - \hat{\mu})' \right]$$

$$\hat{\Phi} = \frac{1}{T}\sum_{t=1}^{T} \left[(R_t - \hat{\mu})(R_t - \hat{\mu})' \otimes (R_t - \hat{\mu})' \right]$$

$$\hat{\Psi} = \frac{1}{T}\sum_{t=1}^{T} \left[(R_t - \hat{\mu})(R_t - \hat{\mu})' \otimes (R_t - \hat{\mu})' \otimes (R_t - \hat{\mu})' \right]$$

FACTOR MODELS AND THE ESTIMATION OF HIGHER ORDER MOMENTS

Linear factor model

- Asset returns are generated by a *K*-dimensional vector of factors and the remaining idiosyncratic term is independent:

$$\begin{pmatrix} r_{1t} \\ r_{2t} \\ \vdots \\ r_{Nt} \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1K} \\ b_{21} & b_{22} & \cdots & b_{2K} \\ & & \ddots & \vdots \\ b_{N1} & b_{N2} & \cdots & b_{NK} \end{pmatrix} \begin{pmatrix} f_{1t} \\ f_{2t} \\ \vdots \\ f_{Kt} \end{pmatrix} + \begin{pmatrix} e_{1t} \\ e_{2t} \\ \vdots \\ e_{Nt} \end{pmatrix}.$$

- Matrix notation: $r_t = a + Bf_t + e_t$

Solution: Impose structure to reduce the number of parameters to estimate

Input:

- History of Returns
- K factors

Assumption of a linear factor model

Covariance matrix Σ:
$N(K+1)+\mathbf{K}(\mathbf{K}+1)/2$ elements

Coskewness matrix Φ:
$N(K+1)+\mathbf{K}(\mathbf{K}+1)(\mathbf{K}+2)/6$ elements

Cokurtosis matrix Ψ:
$N(K+2)+K(K+1)/2+\mathbf{K}(\mathbf{K}+1)(\mathbf{K}+2)(\mathbf{K}+3)/24$ elements

Total number of unique elements:
$N(K+3)+[(1+(K+2)/3+(K+2)(K+3)/12]K(K+1)/2$

	Unrestricted	One factor	Two factors	Three factors
N=5	120	23	37	61
N=20	10,605	83	112	151
N=100	4,598,025	403	512	631

Why?

- Under the factor model, we can rewrite the higher order **comoments as a function of the comoments of the factors and a residual matrix that mostly contains 0s** (because of the independence assumption):

$$\Sigma = BSB' + \Delta,$$

$$\Phi = BG(B' \otimes B') + \Omega,$$

$$\Psi = BP(B' \otimes B' \otimes B') + \Upsilon,$$

- With:

$$S = E\left[(f_t - \mu_f)(f_t - \mu_f)'\right]$$
$$G = E\left[(f_t - \mu_f)(f_t - \mu_f)' \otimes (f_t - \mu_f)'\right]$$
$$P = E\left[(f_t - \mu_f)(f_t - \mu_f)' \otimes (f_t - \mu_f)' \otimes (f_t - \mu_f)'\right]$$

- Estimate of exposure matrix *B*: Equation by equation least squares;
- The residual covariance matrix *Δ* is zero everywhere except on the diagonal:

$$\frac{1}{T}\sum_{t=1}^{T}\hat{\varepsilon}_{it}^{2}$$

- The residual coskewness matrix *Ω* is zero everywhere except for the elements corresponding to the expected third moment of the idiosyncratic terms.

$$\frac{1}{T}\sum_{t=1}^{T}\hat{\varepsilon}_{it}^{3}$$

- The residual cokurtosis matrix *Y* is zero everywhere except for:
  - the kurtosis of one asset [when *i = j = k = l*],

$$6b_i^{'}Sb_i\, E\left[e_{it}^2\right] + E\left[e_{it}^4\right]$$

  - the cokurtosis between two assets [when *(i=j=k)* and *l≠i*]

$$3b_i^{'}Sb_l\, E\left[e_{it}^2\right]$$

  - and the cokurtosis between 3 assets [when *(i = j) ≠* (*k = l)*].

$$b_i^{'}Sb_i\, E\left[e_{kt}^2\right] + b_k^{'}Sb_k\, E\left[e_{it}^2\right] + E\left[e_{it}^2\right]E\left[e_{kt}^2\right]$$

- Code is currently available from:

[https://bitbucket.org/**rossbennett**34/momentestimation/src](https://bitbucket.org/rossbennett34/momentestimation/src)

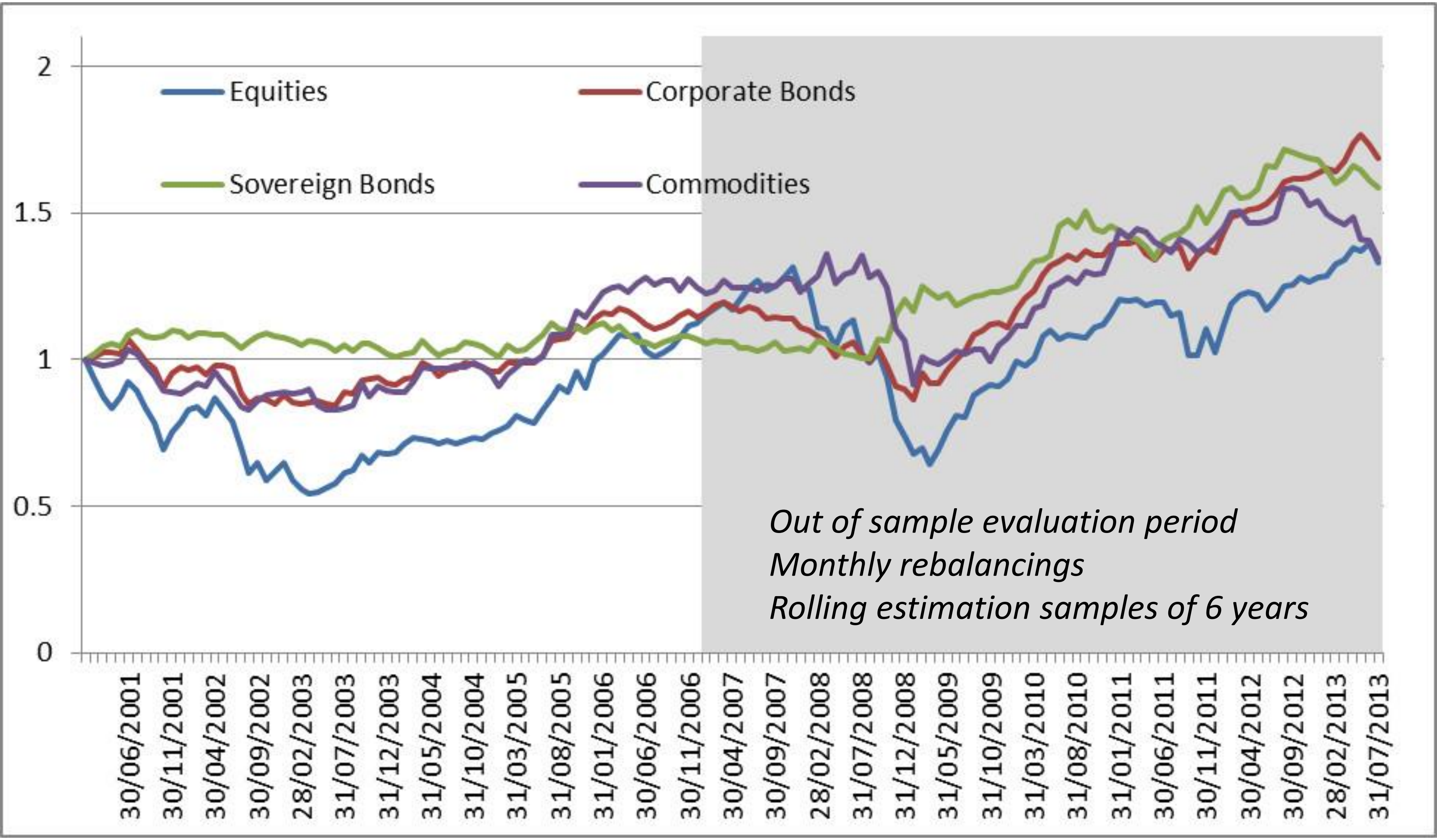
- Nested loops: speed gains through the use of Rcpp

APPLICATION TO THE OPTIMZATION OF A BONDS-EQUITIES-COMMODITIES PORTFOLIO

Application: Asset allocation

17 assets, 2001-2013:

- four **equity** benchmarks (Europe, North America, Pacific and Emerging Markets),
- eight **bond** indices (**corporate** developed high yield index in EUR and USD, corporate developed investment grade index in EUR, corporate emerging investment grade in USD, **sovereign** developed investment grade in USD, EUR, JPY and sovereign developed and emerging in USD)
- five **commodity** indices (agriculture, energy, industrial metals, livestock and precious metals).



Equities
Corporate Bonds
Sovereign Bonds
Commodities
2
1.5
1
0.5
0
Out of sample evaluation period
Monthly rebalancings
Rolling estimation samples of 6 years
30/06/2001
30/11/2001
30/04/2002
30/09/2002
28/02/2003
31/07/2003
31/12/2003
31/05/2004
31/10/2004
31/03/2005
31/08/2005
31/01/2006
30/06/2006
30/11/2006
30/04/2007
30/09/2007
28/02/2008
31/07/2008
31/12/2008
31/05/2009
31/10/2009
31/03/2010
31/08/2010
31/01/2011
30/06/2011
30/11/2011
30/04/2012
30/09/2012
28/02/2013
31/07/2013

- We will consider two objectives:
  - Minimize variance
  - Maximize expected utility (CRRA, setting mean to 0)

$$EU_{\gamma}(w) = -\frac{\gamma}{2} m_{(2)}(w) + \frac{\gamma(\gamma+1)}{6} m_{(3)}(w) - \frac{\gamma(\gamma+1)(\gamma+2)}{24} m_{(4)}(w).$$

- Constraints:
  - Long only, fully invested;
  - Each asset class has to contribute equally to the portfolio risk
    - Variance
    - Or expected shortfall.
- Estimation of portfolio moments using the sample estimator, single factor or 3-factor model (statistical). Rolling estimation sample of 6 years.
- Optimization with DEoptim.

Table A.3: Monthly returns analysis of minimum variance/maximum expected utility strategies under equal variance/ES constraint over the period January 2007-July 2013 (in EUR).

Objective/ Risk Contribution	Estimator	Ann. return	Annualized standard deviation	Skewness	Excess kurtosis	95% Historical VaR	Max Drawdown
Min Variance/	Sample	1.02%	10.69%	-1.60	9.02	-5.60%	-25.52%
Variance	Single factor	3.02%	7.54%	0.49	1.67	-2.89%	-8.99%
	Multifactor	3.48%	7.84%	0.71	1.73	-2.83%	-18.56%
Min Variance/	Sample	5.08%	8.46%	0.17	0.90	-3.38%	-16.45%
E.Shortfall	Single factor	5.54%	7.90%	0.40	0.55	-2.96%	-7.86%
	Multifactor	4.70%	8.35%	0.33	0.52	-3.27%	-12.40%
Max E.Utility/	Sample	3.36%	8.32%	0.06	2.98	-3.27%	-9.89%
Variance	Single factor	5.85%	8.57%	0.64	0.70	-3.04%	-10.26%
$(\gamma = 5)$	Multifactor	4.23%	8.73%	1.06	3.41	-3.44%	-13.27%
Max E.Utility/	Sample	5.47%	7.07%	0.27	0.77	-2.68%	-9.17%
E.Shortfall	Single factor	4.43%	9.59%	-0.88	6.75	-4.40%	-18.24%
$(\gamma = 5)$	Multifactor	5.21%	9.67%	-0.98	6.88	-4.43%	-13.31%
Max E.Utility/	Sample	3.64%	8.03%	0.38	0.29	-3.19%	-11.65%
Variance	Single factor	1.80%	7.97%	0.01	0.57	-3.55%	-11.67%
$(\gamma = 10)$	Multifactor	4.53%	8.90%	0.27	-0.18	-3.59%	-11.87%
Max E.Utility/	Sample	4.82%	9.05%	-0.05	0.38	-3.86%	-11.99%
E.Shortfall	Single factor	4.49%	8.25%	0.26	0.91	-3.28%	-10.27%
$(\gamma = 10)$	**Multifactor**	**9.47%**	**9.35%**	**1.43**	**3.32**	**-2.25%**	**-8.06%**

Main result:

- Three actions seem to increase the out-of-sample return and reduce the portfolio downside risk:

(i) imposing more structure on the estimates (moving from the sample estimator to the factor model based estimates);

(ii) switching from an equal variance contribution constraint to an equal expected shortfall constraint;

(iii) switching from a minimum variance objective to a CRRA expected utility objective.

- This is for one sample…
- More research is needed in terms of sensitivity to the sample studied, turnover analysis, risk budgets, the treatment of currency effects, the handling of outliers and, importantly, the choice of factors.

CONCLUSION

Conclusion

- Higher order moments are important in portfolio allocation;
- Notoriously difficult to estimate already in moderate dimension because of curse of dimensionality;
- Structure needs to be imposed in order to reduce the number of parameters to estimate;
- This paper studies the use of the multifactor model: many parameters are zero ➔ a huge reduction in number of parameters to estimate.

	Unrestricted	One factor	Two factors	Three factors
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Current research

- The multifactor model assumption solves problems, but raises others:
  - How to select factors; especially those that are informative about the higher order comoments (not just PCA explaining the covariance);
  - Impact of extremes: Winsorization accounting for the non-normality;
  - Dynamics;
  - Model diagnostics:

Model diagnostics: Are the elements really 0?

- The fact that many of these elements are 0 is a model assumption (idiosyncratic terms are independent).
- In current research we develop a test for this.
  - Good news is: asymptotic distribution of test statistics based on products of residuals of different assets are independent of the equation by equation regression estimation needed to compute those residuals (Randles, 1997)

$$\frac{1}{T}\sum_{t=1}^{T}\hat{\varepsilon}_{it}\hat{\varepsilon}_{jt};\frac{1}{T}\sum_{t=1}^{T}\hat{\varepsilon}_{it}\hat{\varepsilon}_{jt}\hat{\varepsilon}_{kt};\frac{1}{T}\sum_{t=1}^{T}\hat{\varepsilon}_{it}\hat{\varepsilon}_{jt}\hat{\varepsilon}_{kt}\hat{\varepsilon}_{lt}$$

$$i<j;i<j<k;i<j<k<l$$

  - Challenge is: so many elements that can be tested; individual test (many spurious detections); joint hypothesis test (curse of dimensionality).

References

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