Efficient Multivariate Analysis of Change Points

The e-cp3o procedure

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Change Point Analysis

- Partition time series into homogeneous segments.
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  - Break points
  - Breakouts
  - Regime changes
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Difficulties

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- Difficult even if number of change points is known.
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- Number of change points is unknown.

- Unknown distance between change points.

- Difficult even if number of change points is known.
  - Exponential in number of change points.
Divergence Measure

- Let $X_1$ and $Y_1$ be independent random vectors. And $(X_2, Y_2)$ and iid copy of $(X_1, Y_2)$. 
Divergence Measure

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Divergence Measure

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• The energy distance between the distributions of $X_1$ and $Y_1$

$$
\mathcal{E}(X_1, Y_1|\alpha) = 2E|X_1 - Y_1|^\alpha - E|X_1 - X_2|^\alpha - E|Y_1 - Y_2|^\alpha
$$

$$= 2 \cdot \text{between} - X \text{ within} - Y \text{ within}
$$
Divergence Measure

Let $X_n = \{x_1, x_2, \ldots, x_n\}$ and $Y_m = \{y_1, y_2, \ldots, y_m\}$. 
Divergence Measure

Let $X_n = \{x_1, x_2, \ldots, x_n\}$ and $Y_m = \{y_1, y_2, \ldots, y_m\}$.

Their sample divergence is given by

$$\hat{E}(X_n, Y_m|\alpha) = \frac{2}{mn} \sum_{i,j} |x_i - y_j|^\alpha$$

$$- \binom{n}{2}^{-1} \sum_{i<j} |x_i - x_j|^\alpha$$

$$- \binom{m}{2}^{-1} \sum_{i<j} |y_i - y_j|^\alpha.$$
Finding Change Points

Let $k > 0$ and

$$\hat{R}(X_n, Y_m | \alpha) = \frac{mn}{(m + n)^2} \hat{E}(X_n, Y_m | \alpha).$$
Finding Change Points

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$$\hat{R}(X_n, Y_m|\alpha) = \frac{mn}{(m+n)^2} \hat{E}(X_n, Y_m|\alpha).$$

The best partitioning of time series $Z_1, Z_2, \ldots, Z_T \in \mathbb{R}^d$, with $k$ change points is given

$$\beta_k(T) = \max_{\tau_1, \tau_2, \ldots, \tau_k} \hat{R}(C_0, C_1|\alpha) + \hat{R}(C_1, C_2|\alpha) + \cdots + \hat{R}(C_{k-1}, C_k|\alpha).$$
Finding Change Points

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With $C_i = \{Z_{\tau_i-1}+1, Z_{\tau_i-1}+2, \ldots, Z_{\tau_i}\}$
Finding Change Points

Let $k > 0$ and

$$\hat{R}(X_n, Y_m | \alpha) = \frac{mn}{(m + n)^2} \hat{E}(X_n, Y_m | \alpha).$$

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With $C_i = \{Z_{\tau_i-1+1}, Z_{\tau_i-1+2}, \ldots, Z_{\tau_i}\}$

Can find $\hat{\tau}_1, \ldots, \hat{\tau}_k$ in $O(kT^3)$ time.
Finding Change Points

Finding change points using $\hat{R}$ is too slow.
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Finding change points using $\hat{R}$ is too slow.
Instead use

$$\tilde{\mathcal{E}}(X_n, Y_m|\alpha, \delta) = \frac{2}{\# B^\delta} \sum_{(i,j) \in B^\delta} |x_i - y_j|^\alpha - \frac{1}{\# W_x^\delta} \sum_{(i,j) \in W_x^\delta} |x_i - x_j|^\alpha - \frac{1}{\# W_y^\delta} \sum_{(i,j) \in W_y^\delta} |y_i - y_j|^\alpha.$$
Finding Change Points

Instead use

\[
\tilde{E}(X_n, Y_m|\alpha, \delta) = \frac{2}{\# B_\delta} \sum_{(i,j) \in B_\delta} |x_i - y_j|^{\alpha} - \frac{1}{\# W_{x_\delta}} \sum_{(i,j) \in W_{x_\delta}} |x_i - x_j|^{\alpha} - \frac{1}{\# W_{y_\delta}} \sum_{(i,j) \in W_{y_\delta}} |y_i - y_j|^{\alpha}.
\]

Sets $B_\delta$, $W_{x_\delta}$, and $W_{y_\delta}$ are sets of index pairs.
Finding Change Points

Using $\tilde{R}$ allows us to find $\hat{\tau}_1, \ldots, \hat{\tau}_k$ in $O(kT^2)$ time.
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Still one significant negative property of this approach:
Finding Change Points

Using $\tilde{\mathcal{K}}$ allows us to find $\hat{\tau}_1, \ldots, \hat{\tau}_k$ in $O(kT^2)$ time.

Still one significant negative property of this approach:
- performs many unnecessary calculations
Finding Change Points

Remove points from the search space that have probability less than \( \epsilon \) of being a true change point.
Finding Change Points

Remove points from the search space that have probability less than $\epsilon$ of being a true change point.
Finding Change Points

Bound the change created by an additional change point.

\[ \tilde{R}(Z_{t+1}, Z_{u+1} | \alpha) - \tilde{R}(Z_{s+1}, Z_{u+1} | \alpha) - \tilde{R}(Z_{s+1}, Z_{u+1} | \alpha) < \Gamma \]
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$$\tilde{\mathcal{R}}(Z_{v+1}, Z_{t+1} | \alpha) - \tilde{\mathcal{R}}(Z_{v+1}, Z_{t+1} | \alpha) - \tilde{\mathcal{R}}(Z_{t+1}, Z_{s+1} | \alpha) < \Gamma$$
Finding Change Points

Bound the change created by an additional change point.

\[ \tilde{R}(Z_{v+1}, Z_{t+1}^u | \alpha) - \tilde{R}(Z_{v+1}, Z_{t+1}^s | \alpha) - \tilde{R}(Z_{t+1}, Z_{s+1}^u | \alpha) < \Gamma \]
Finding Change Points

Bound the change created by an additional change point.

$$\tilde{R}(Z_{v+1}^t, Z_{t+1}^u|\alpha) - \tilde{R}(Z_{v+1}^t, Z_{s+1}^s|\alpha) - \tilde{R}(Z_{t+1}^t, Z_{s+1}^u|\alpha) < \Gamma$$
Finding Change Points

Unknown distributions make finding $\Gamma$ impossible.
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Unknown distributions make finding $\Gamma$ impossible.

Consider a probabilistic version

$$\mathbb{P}\left(\tilde{\mathcal{R}}(Z_{v+1}^t, Z_{t+1}^u|\alpha) - \tilde{\mathcal{R}}(Z_{v+1}^t, Z_{t+1}^s|\alpha) - \tilde{\mathcal{R}}(Z_{t+1}^s, Z_{s+1}^u|\alpha) \geq \Gamma \epsilon \right) \leq \epsilon$$
Finding Change Points

Unknown distributions make finding $\Gamma$ impossible.

Consider a probabilistic version

$$P \left( \tilde{\mathcal{R}}(Z_{v+1}^{t}, Z_{t+1}^{u} | \alpha) - \tilde{\mathcal{R}}(Z_{v+1}^{t}, Z_{s+1}^{s} | \alpha) - \tilde{\mathcal{R}}(Z_{t+1}^{s}, Z_{s+1}^{u} | \alpha) \geq \Gamma \epsilon \right) \leq \epsilon$$

Remove $t$ from the search space if

$$\zeta_{k}(t) + \tilde{\mathcal{R}}(Z_{v+1}^{t}, Z_{t+1}^{s} | \alpha) + \Gamma \epsilon < \zeta_{k}(s)$$
Using e-cp3o

The e-cp3o algorithm is currently available in the ecp package.
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e.cp3o(Z, K, delta, alpha, eps)
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- \( Z \): Time series as a matrix
Using e-cp3o

The e-cp3o algorithm is currently available in the ecp package.

e.cp3o(Z, K, delta, alpha, eps)

- Z: Time series as a matrix
- K: Maximum number of change points to fit
Using e-cp3o

The e-cp3o algorithm is currently available in the `ecp` package.

\[ e\text{-}cp3o(Z, K, \delta, \alpha, \epsilon) \]

- **Z**: Time series as a matrix
- **K**: Maximum number of change points to fit
- **\delta**: Minimum number of observations between change points
Using e-cp3o

The e-cp3o algorithm is currently available in the ecp package.

e.cp3o(Z, K, delta, alpha, eps)

- Z: Time series as a matrix
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- delta: Minimum number of observations between change points
- alpha: Distance weighting
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\[ \text{e.cp3o}(Z, K, \delta, \alpha, \epsilon) \]

- **Z**: Time series as a matrix
- **K**: Maximum number of change points to fit
- **\delta**: Minimum number of observations between change points
- **\alpha**: Distance weighting
- **\epsilon**: Pruning probability
Look for change in the price of gold (USD) from January 1, 2000 to January 1, 2015. This results in 3789 observations.
Gold Price (USD)

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Obtain data using Quandl.
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Obtain data using **Quandl**.

```r
> library("Quandl")
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Obtain data using Quandl.

```r
> library("Quandl")
> gold = Quandl("BUNDESBANK/BBK01_WT5511", type="xts", transformation="rdiff", trim_start="1999-12-31", trim_end="2015-01-01")
```
Use `quantmod` to create time series plot.
Gold Price (USD)

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> library("quantmod")
```
Gold Price (USD)

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```r
> library("quantmod")
> chartSeries(gold, theme=chartTheme("white"))
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![Gold Price Time Series Plot](image)
Gold Price (USD)

Use **ecp** package to find change points.
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```r
> library("ecp")
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```r
> library("ecp")
> res = e.cp3o(Z=gold, K=20, delta=6, alpha=1, eps=0.01)
```

Gold Price (USD)
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> res$time
```

17/22
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[1] 10.866
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> index(gold[res$estimates])
```

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[1] 10.866
> index(gold[res$estimates])
  "2011-09-29" "2013-04-17"
```
Gold Price (USD)
Amazon & Ebay

Look for changes returns for Amazon and Ebay stock, from January 1999 to January 2015. This time series has 4038 observations.

```python
> ebay = Quandl("GOOG/NASDAQ_EBAY", transformation="rdiff", trim_start="1999-01-01", trim_end="2014-12-31")
> amazon = Quandl("GOOG/NASDAQ_AMZN", transformation="rdiff", trim_start="1999-01-01", trim_end="2014-12-31")
> prices = matrix(c(ebay$Close, amazon$Close), ncol=2, byrow=T)
```
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                trim_end="2014-12-31")

> prices = matrix(c(ebay$Close,
                   amazon$Close),ncol=2,byrow=T)
```

19/22
> res = e.cp3o(Z=prices, K=20, delta=6, alpha=1, eps=0.01)
Amazon & Ebay

\[
\begin{align*}
> \text{res} &= \text{e.cp3o}(Z=\text{prices}, K=20, \text{delta}=6, \alpha=1, \\
&\quad \text{eps}=0.01) \\
>
\text{res$\text{time}}
\end{align*}
\]
Amazon & Ebay

```r
> res = e.cp3o(Z=prices, K=20, delta=6, alpha=1, eps=0.01)
> res$time
[1] 22.142
```
Amazon & Ebay

\[
\text{> res = e.cp3o}(Z=\text{prices}, K=20, \text{delta}=6, \text{alpha}=1, \epsilon=0.01)
\]

\[
\text{> res$\text{time}}
\]
\[
[1] \hspace{0.5cm} 22.142
\]

\[
\text{> ebay$\text{Date}[\text{res$estimates}]
\]
Amazon & Ebay

> res = e.cp3o(Z=prices, K=20, delta=6, alpha=1, eps=0.01)

> res$time
[1] 22.142

> ebay$Date[res$estimates]
   "1999-12-28"
Amazon & Ebay

1/5/99 - 6/6/00

6/7/00 - 12/15/03

12/16/03 - 1/9/08

1/10/08 - 12/1/11

12/2/11 - 12/31/14
Bibliography I


