Inefficiency of Modified VaR and ES

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29 May 2015

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[1] Joint work with Dr Douglas Martin

Standard Error of Parametric VaR and ES

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- If $\hat{\mu}, \hat{\sigma}$ are MLE then $g\left(\hat{\mu}, \hat{\sigma}^2\right)$ is MLE and $\text{var}\left(g\left(\hat{\mu}, \hat{\sigma}^2\right)\right)$ is the smallest achievable variance in large samples

Modified VaR and ES

Modified VaR (Zangari, 1996)

$$mVaR_\gamma = \mu + \sigma g_\gamma, \quad g_\gamma = z_\gamma + \frac{Sk}{6}\left(z_\gamma^2 - 1\right) + \frac{\kappa}{24}\left(z_\gamma^3 - 3z_\gamma\right) - \frac{s_k^2}{36}\left(2z_\gamma^3 - 5z_\gamma\right)$$

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Modified ES (Boudt, Peterson, and Croux, 2008)

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- $mVaR_\gamma$, $mES_\gamma$are not MLE so thier variance will be larger than the variance of MLE estimator

$$ARE\left(\hat{\theta}, g\left(\hat{\mu}, \hat{\sigma}^2\right)\right) = \frac{V_{g(\hat{\mu}, \hat{\sigma}^2)}}{V_{\hat{\theta}}} \leq 1$$

Asymptotic Variance of Modified Var and ES

- We want to calculate the variance of the modifed risk estimate, $\hat{\theta}$

$$\hat{\theta}-\theta = (\hat{\mu}-\mu)+C_0(\hat{\sigma}-\sigma)+C_1\left(\hat{\sigma}\hat{sk}-\sigma s_k\right)+C_2(\hat{\sigma}\hat{\kappa}-\sigma\kappa)-C_3\left(\hat{\sigma}\hat{sk}^2-\sigma s_k^2\right)$$

$$\text{var}\left[\hat{\theta}-\theta\right] = \boldsymbol{x}^\top \boldsymbol{\Sigma}_\theta \boldsymbol{x}, \qquad x = \begin{bmatrix} 1 & C_0 & C_1 & C_2 & -C_3 \end{bmatrix}^\top$$

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- (Serfling, 1980), gives the asymptotic covariance of the first 4 central moments

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma^2 & & & \\ \mu_3 & \mu_4-\sigma^4 & & \\ \mu_4 & \mu_5-4\sigma^2\mu_3 & \mu_6-\mu_3^2-6\sigma^2\mu_4+9\sigma^6 & \\ \mu_5 & \mu_6-\sigma^2\mu_4-4\mu_3^2 & \mu_7-5\mu_3\mu_4-3\sigma^2\mu_5+12\sigma^4\mu_3 & \mu_8-\mu_4^2-8\mu_3\mu_5+16\sigma_2\mu_3^2 \end{bmatrix}$$

Procedure to compute elements of $\boldsymbol{\Sigma}_\theta$

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  - Consider $z = x + y$.
  - Use the multivariate delta method to compute $\text{var}(z)$
  - Use the variance of the diagonal elements $\text{var}(x)$ and $\text{var}(y)$ to calculate $\text{cov}(x, y) = \frac{1}{2}\left[\text{var}(z) - \text{var}(x) - \text{var}(y)\right]$

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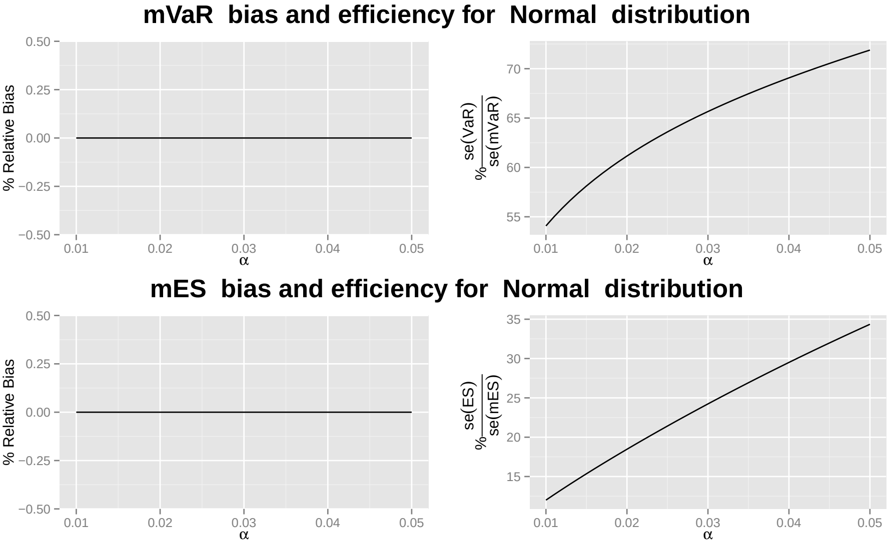
Results for Normal distribution

https://github.com/arorar/VarInModRisk



Results for Standardized Student-t distribution

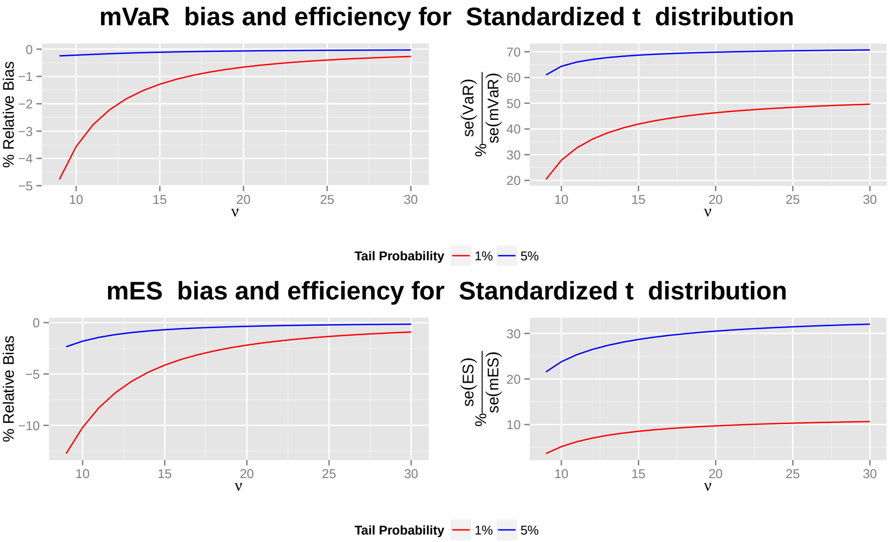


Figure: Asymptotics

Small sample results for Standardized Student-t distribution

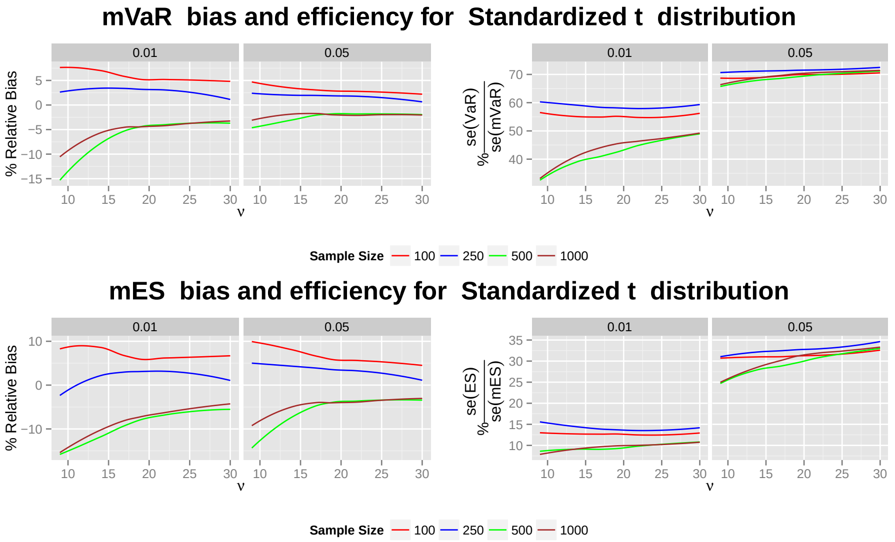


Figure: Small sample behavior

References

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