QUANTITATIVE ANALYSIS OF DUAL MOVING AVERAGE INDICATORS IN AUTOMATED TRADING SYSTEMS

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INTRODUCTION
Dual Moving Average Technical Trading Strategy [Jaekle and Tomasini(2009)]

Long position - fast MA above slow MA

Short position - fast MA below slow MA
Luxor dual moving average trading strategy
Luxor historically analyzed via inferential statistics

- Discrete time setting
- Historical data [Jaekle and Tomasini(2009)]
- Bootstrapped resampled [Efron(1979)] [William Brock(1992)]

to determine

- Expected returns
- Draw down
- Select strategy parameters etc.

Our goal
Assume price dynamics are controlled by a stochastic differential equation (SDE) and derive the Luxor closed form expected log returns in a continuous time setting

- Inspired by work in options pricing [Black and Scholes(1973), Merton(1973)]
- Gain analytical tractability by using the natural logarithm of the price
- Build frame work for analyzing trading strategies
Assumptions
To gain analytical tractability we assume a frictionless market

1. Difference between the bid and ask price is infinitesimal
2. Transaction costs are zero
3. Transaction execution is instantaneous
4. Buy (sell) transactions fully execute at the current ask (bid) price
5. Market contains a single risky asset with price $S_t$
Continuous Time Model
Continuous Time Model

Price Dynamics

- Let \( \left( \Omega, \mathcal{F}, \{ \mathcal{F}_t \}_{t \geq 0}, \mathbb{P} \right) \) be a complete filtered probability space.
- Dynamics of \( S_t \) under \( \mathbb{P} \) is controlled by the SDE
  \[
  dS_t = \mu S_t \, dt + \sigma S_t \, dW_t \quad \mu, \sigma \in \mathbb{R} \text{ constant}
  \]
- \( W_t \) is one dimensional Brownian motion whose history up to time \( t \) contained in the filtration \( \{ \mathcal{F}_t \}_{t \geq 0} \).
- For each \( \omega \in \Omega \) there exits a function
  \[
  W_t(\omega) : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+ \quad (\text{the samples})
  \]
- \( W_0(\omega) = 0 \), disjoint increments of \( W_t(\omega) \) are independent, increments \( W_{\Delta t}(\omega) \) are normally distributed with mean zero and variance \( \Delta t \), and \( W_t(\omega) \) is continuous on \([0, t]\).
SDE Solution
Let

\[ f(s) = \ln s, \quad f'(s) = \frac{1}{s}, \quad f''(s) = -\frac{1}{s^2} \]

\[ (dt)^2 = 0, \quad (dt)(dW_t) = 0, \quad (dW_t)^2 = dt \quad ([W, W]_t = t) \]

Apply Itô’s formula [Shreve(2004)] to \(d \ln S_t\) where \(dS_t = \mu S_t dt + \sigma S_t dW_t\)

\[ df(S_t) = f'(S_t) dS_t + \frac{1}{2} f''(S_t) (dS_t)^2 \]

\[ d \ln S_t = \frac{1}{S_t} dS_t + \frac{1}{2} \left(-\frac{1}{S_t^2}\right) (dS_t)^2 \]

\[ = \frac{1}{S_t} (\mu S_t dt + \sigma S_t dW_t) + \frac{1}{2} \left(-\frac{1}{S_t^2}\right) (\sigma^2 S_t^2 dt) \]

\[ = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dW_t \]  

(1)
**SDE Solution**

Integrate equation 1 from 0 to \( t \) and apply exponential function to both sides

\[
\int_0^t d \ln S_t = \int_0^t \left( \mu - \frac{\sigma^2}{2} \right) \, du + \int_0^t \sigma \, dW_u
\]

\[
\ln S_t = \ln S_0 + \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma W_t
\]

\[
S_t = S_0 \, e^{\left( \mu - \frac{\sigma^2}{2} \right) t + \sigma W_t}
\]

The solution \( S_t \) is known as geometric Brownian motion.
Model

Dividing both sides of the SDE by $S_t$, it can be rewritten as

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$$

(3)

In essence, the instantaneous return is a constant term plus a volatility term.

- If we knew that either $\mu >> \frac{\sigma^2}{2}$ or $\mu << \frac{\sigma^2}{2}$ and the SDE model is accurate, investing would be easy; thus, we assume $|\mu - \frac{\sigma^2}{2}|$ is not large and $\mu$ is not easy to determine for a given asset over a short time frame.
Exponentially Filtered Stochastic Process

An exponentially filtered stochastic process $\Upsilon_t[X_t; \tau]$ where $\tau > 0$ is a convolution of the exponential kernel function $\frac{1}{\tau} e^{-\frac{1}{\tau} t}$ with the stochastic process $X_t$.

$$\Upsilon_t[X_t; \tau] = \int_0^t \frac{1}{\tau} e^{-\frac{1}{\tau} (t-u)} X_u du$$  \hspace{1cm} (4)

Note:

- $\Upsilon_t[X_t; \tau]$ is a path dependent time series operator
- $\Upsilon_t[X_t; \tau]$ is $\{\mathcal{F}_t\}_{t \geq 0}$ measurable - can calculate $\Upsilon_t[X_t; \tau]$ given filtration at time $t$
- Lower limit is usually defined as $-\infty$; however, we work with processes on $[0, T]$ and assume a steady state is reached in finite time at $t > t_{ss}$
Strategy Definition
Strategy Definition

Strategy Components [Peterson(2015)]

• Filters - select instruments
• Indicators - quantitative values derived from market data
• Signals - respond to interactions between filters, market data, and indicators
• Rules - make path dependent actionable decisions when signals fire

Formulate Luxor as a set of equations following this model
**Difference Indicator**

The difference indicator $\Psi_t : \mathbb{R} \rightarrow \mathbb{R}$ is the difference of two exponential filters applied to a stochastic process $X_t$ where $0 < \tau_1 < \tau_2$.

$$\Psi_t[X_t; \tau_1, \tau_2] = \gamma_t[X_t; \tau_1] - \gamma_t[X_t; \tau_2]$$

- Use natural logarithm of the asset price $X_t = \ln S_t$ to gain analytical tractability.
- $\Psi_t[X_t; \tau_1, \tau_2]$ known as moving average convergence divergence (MACD) oscillator in technical analysis.
Position Indicator

The position indicator $\varrho_t : \mathbb{R} \rightarrow \{-1, 0, 1\}$ is the composition of the signum function with the difference indicator and indicates potential long or short position opportunities.

\[ \varrho_t = (\text{sgn} \circ \Psi)_t = \text{sgn}(\Psi_t) = \begin{cases} 
1 & \Psi_t \geq 0 \quad \text{long} \\
0 & \Psi_t = 0 \quad \text{previous} \\
-1 & \Psi_t < 0 \quad \text{short} 
\end{cases} \]
**Strategy Definition**

**Position Entry/Exit Signals**

Position indicator transitions

- \((-1 \rightarrow +1)\) or \((-1 \rightarrow 0 \rightarrow +1)\) exit any short position and enter a long position
- \((1 \rightarrow -1)\) or \((1 \rightarrow 0 \rightarrow -1)\) exit any long position and enter short position

Initial entry/final exit occur on first position indicator entry/exit signal after steady state is reached at \(t > t_{ss}\)
Strategy Definition

Rules
Always act on position indicator signals (identity function)

• Gain analytical tractability
• Efficient implementation uses rules to define
  • When and how to enter or exit the market
  • Determine position size
  • Manage risk
EVALUATION CRITERIA
**Luxor Log Return**

The log return $r_t$ of the Luxor strategy on $[0, t]$ is given by the integral of the position indicator $\varrho$ with respect to the change of the logarithm of the asset price.

$$ r_t = \int_0^t \varrho_u d\ln S_u $$

(6)

Assumes equal size positions in the asset
Example
Strategy Analysis
**Strategy Analysis**

**Exponentially Filtered Logarithm of Asset Price**

Construct the exponentially filtered logarithm of the asset price \( \Upsilon_t[\ln S_t, \tau] \) by substituting the logarithm of the SDE solution \( S_t \) from equation 2 for \( X_t \) in the filter definition in equation 4.

\[
\Upsilon_t[\ln S_t, \tau] = \int_0^t \frac{1}{\tau} e^{-\frac{1}{\tau}(t-u)} \left[ \ln S_0 + \left( \mu - \frac{\sigma^2}{2} \right) u + \sigma W_u \right] du
\]

\[
= \frac{e^{-\frac{1}{\tau}t}}{\tau} \left[ \ln S_0 \int_0^t e^{\frac{1}{\tau}u} du + \left( \mu - \frac{\sigma^2}{2} \right) \int_0^t u e^{\frac{1}{\tau}u} du + \sigma \int_0^t e^{\frac{1}{\tau}u} W_u du \right]
\]

(7)

First two integrals in \( \Upsilon_t[\ln S_t, \tau] \) are deterministic Riemann integrals wrt time and have explicit solutions. The last integral is a stochastic Riemann integral of Brownian motion wrt time.
Exponentially Filtered Logarithm of Asset Price

First integral in $\gamma_t[\ln S_t, \tau]$, equation is solved using the fundamental theorem of calculus.

\[
\int_0^t e^{\frac{1}{\tau}u} du = \tau \left[ e^{\frac{1}{\tau}u} \right]_0^t = \tau \left( e^{\frac{1}{\tau}t} - 1 \right) \tag{8}
\]

The second integral requires in addition to the fundamental theorem of calculus integration by parts where $\int x dy = xy - \int y dx$. Let $x = u$ which implies $dx = du$, and let

\[
dy = e^{\frac{1}{\tau}u} du \\
y = \int e^{\frac{1}{\tau}u} du = \tau e^{\frac{1}{\tau}u} \tag{9}
\]
Exponentially Filtered Logarithm of Asset Price

Using the results from equation 9 to complete the integration by parts solves the second integral in $\Upsilon_t[\ln S_t, \tau]$ from equation 7.

\[
\int_0^t u e^{t/u} du = \tau \left[ u e^{t/u} \right]_0^t - \int_0^t \tau e^{t/u} du
\]
\[
= \tau t e^{1/\tau} - \tau^2 \left[ e^{1/\tau} \right]_0^t
\]
\[
= \tau \left[ te^{1/\tau} - \tau \left( e^{1/\tau} - 1 \right) \right]
\]

(10)
Exponentially Filtered Logarithm of Asset Price

Inserting the results of equations 8 and 10 into \( \Upsilon_t[\ln S_t, \tau] \), equation 7, results in an equation with two deterministic terms and a stochastic term which is a Riemann integral of scaled Brownian motion wrt time.

\[
\Upsilon_t[\ln S_t, \tau] = \frac{e^{-\frac{1}{\tau}t}}{\tau} \left[ \tau \ln S_0 \left( e^{\frac{1}{\tau}t} - 1 \right) 
+ \tau \left( \mu - \frac{\sigma^2}{2} \right) \left[ t e^{\frac{1}{\tau}t} - \tau \left( e^{\frac{1}{\tau}t} - 1 \right) \right] + \sigma \int_0^t e^{\frac{1}{\tau}u} W_u du \right]
\]

\[
= \ln S_0 \left( 1 - e^{-\frac{1}{\tau}t} \right)
+ \left( \mu - \frac{\sigma^2}{2} \right) \left[ t - \tau \left( 1 - e^{-\frac{1}{\tau}t} \right) \right] + \frac{\sigma e^{-\frac{1}{\tau}t}}{\tau} \int_0^t e^{\frac{1}{\tau}u} W_u du \tag{11}
\]
Difference Indicator

Let $0 < \tau_1 < \tau_2$, the difference indicator $\Psi_t[\ln S_t, \tau_1, \tau_2]$ is formed from the difference of two versions of $\Upsilon$ with different values of $\tau$ applied to $\ln S_t$.

$$\Psi_t[\ln S_t, \tau_1, \tau_2] = \Upsilon_t[\ln S_t, \tau_1] - \Upsilon_t[\ln S_t, \tau_2]$$

$$= \ln S_0 \left( 1 - e^{-\frac{1}{\tau_1} t} \right) + \left( \mu - \frac{\sigma^2}{2} \right) \left[ t - \tau_1 \left( 1 - e^{-\frac{1}{\tau_1} t} \right) \right] + \frac{\sigma e^{-\frac{1}{\tau_1} t}}{\tau_1} \int_0^t e^{\frac{1}{\tau_1} u} W_u du$$

$$- \ln S_0 \left( 1 - e^{-\frac{1}{\tau_2} t} \right) - \left( \mu - \frac{\sigma^2}{2} \right) \left[ t - \tau_2 \left( 1 - e^{-\frac{1}{\tau_2} t} \right) \right] - \frac{\sigma e^{-\frac{1}{\tau_2} t}}{\tau_2} \int_0^t e^{\frac{1}{\tau_2} u} W_u du$$

$$= \ln S_0 \left( e^{-\frac{1}{\tau_2} t} - e^{-\frac{1}{\tau_1} t} \right) + \left( \mu - \frac{\sigma^2}{2} \right) \left[ \tau_2 \left( 1 - e^{-\frac{1}{\tau_2} t} \right) - \tau_1 \left( 1 - e^{-\frac{1}{\tau_1} t} \right) \right]$$

$$+ \sigma \int_0^t \left( \frac{1}{\tau_1} e^{-\frac{1}{\tau_1} (t-u)} - \frac{1}{\tau_2} e^{-\frac{1}{\tau_2} (t-u)} \right) W_u du$$

(12)

From this point forward $\Psi_t = \Psi_t[\ln S_t, \tau_1, \tau_2]$. 

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Convert to Itô Integral

When $f_t$ is deterministic, the Itô product rule [Oksendal(2000)], reduces to

$$d(f_t W_t) = (df_t)W_t + f_t dW_t + (df_t)(dW_t)$$

$$= f'_t dt W_t + f_t dW_t$$

Rearranging terms and integrating both sides from 0 to $t$ gives an integration by parts formula that converts a Riemann integral of Brownian motion wrt time to a scaled Brownian motion term and an Itô integral

$$
t_t W_t dt = d(f_t W_t) - f_t dW_t
$$

$$
\int_0^t f'_u W_u du = \int_0^t d(f_u W_u) - \int_0^t f_u dW_u
$$

$$
\int_0^t f'_u W_u du = [f_u W_u]_0^t - \int_0^t f_u dW_u
$$

(13)
Convert to Itô Integral

Let

\[
\frac{df_u}{du} = \sigma \left( \frac{1}{\tau_1} e^{-\frac{1}{\tau_1} (t-u)} - \frac{1}{\tau_2} e^{-\frac{1}{\tau_2} (t-u)} \right)
\]

\[
\int df_u = \sigma \int \left( \frac{1}{\tau_1} e^{-\frac{1}{\tau_1} (t-u)} - \frac{1}{\tau_2} e^{-\frac{1}{\tau_2} (t-u)} \right) du
\]

\[
f_t = \sigma \left( e^{-\frac{1}{\tau_1} (t-u)} - e^{-\frac{1}{\tau_2} (t-u)} \right)
\]

(14)

Rearrange terms as in equation 13

\[
\sigma \int_0^t \left( \frac{1}{\tau_1} e^{-\frac{1}{\tau_1} (t-u)} - \frac{1}{\tau_2} e^{-\frac{1}{\tau_2} (t-u)} \right) W_u du
\]

\[
= \sigma \left[ \left( e^{-\frac{1}{\tau_1} (t-u)} - e^{-\frac{1}{\tau_2} (t-u)} \right) W_u \right]_0^t - \sigma \int_0^t \left( e^{-\frac{1}{\tau_1} (t-u)} - e^{-\frac{1}{\tau_2} (t-u)} \right) dW_u
\]

\[
= \sigma \int_0^t \left( e^{-\frac{1}{\tau_2} (t-u)} - e^{-\frac{1}{\tau_1} (t-u)} \right) dW_u
\]

(15)
**Itô Process**

Substituting the results of the integration by parts into equation 12 converts $\Psi_t$ to an Itô process

$$\Psi_t = \ln S_0 \left( e^{-\frac{1}{\tau_2}t} - e^{-\frac{1}{\tau_1}t} \right)$$

$$+ \left( \mu - \frac{\sigma^2}{2} \right) \left( \tau_2 \left( 1 - e^{-\frac{1}{\tau_2}t} \right) - \tau_1 \left( 1 - e^{-\frac{1}{\tau_1}t} \right) \right)$$

$$+ \sigma \int_0^t \left( e^{-\frac{1}{\tau_2}(t-u)} - e^{-\frac{1}{\tau_1}(t-u)} \right) dW_u$$

(16)
Transient Behavior

Define the first two terms of $\Psi_t$ equation 16 as

\[ b_{t,1} = \ln S_0 \left[ e^{-\frac{1}{\tau_2} t} - e^{-\frac{1}{\tau_1} t} \right] \]  

\[ b_{t,2} = \left( \mu - \frac{\sigma^2}{2} \right) \left[ \tau_2 \left( 1 - e^{-\frac{1}{\tau_2} t} \right) - \tau_1 \left( 1 - e^{-\frac{1}{\tau_1} t} \right) \right] \]
Strategy Analysis: Difference Indicator

Plot of $b_{t,1}$ and $b_{t,2}$ with $\mu = 0$, $\sigma = 0.1$, $\tau_1 = 5$, and $\tau_2 = 10$
**Steady State Assumptions**

We assume that for $S_0 > 1$ and $0 < \tau_1 < \tau_2$ there exists a $t_{ss}$ such that for all $t > t_{ss}$ the transient behavior of the deterministic terms $b_{t,1}$ and $b_{t,2}$ is insignificant to the analysis and can be ignored.

\[
\lim_{t \to \infty} [b_{t,1} + b_{t,2}]
= \lim_{t \to \infty} \left[ \ln S_0 \left( e^{-\frac{1}{\tau_2} t} - e^{-\frac{1}{\tau_1} t} \right) + \left( \mu - \frac{\sigma^2}{2} \right) \left[ \tau_2 \left( 1 - e^{-\frac{1}{\tau_2} t} \right) - \tau_1 \left( 1 - e^{-\frac{1}{\tau_1} t} \right) \right] \right]
= \left( \mu - \frac{\sigma^2}{2} \right) (\tau_2 - \tau_1)
\]

(19)

We assume that

\[
\left| \lim_{t \to \infty} [b_{t,1} + b_{t,2}] - [b_{t,1} + b_{t,2}]_{t=t_{ss}} \right| < \delta
\]

(20)

for some sufficiently small $\delta$. 
Steady State Difference Indicator

\[ \Psi_t = \left( \mu - \frac{\sigma^2}{2} \right) (\tau_2 - \tau_1) + \sigma \int_0^t \left( e^{-\frac{1}{\tau_2} (t-u)} - e^{-\frac{1}{\tau_1} (t-u)} \right) dW_u \]

(21)
Expected Value and Variance of $\Psi_t$

Let $M_t$ represent the Itô integral contained in the definition of $\Psi_t$ in equation 21.

$$
M_t = \sigma \int_0^t \left( e^{-\frac{1}{\tau_2}(t-u)} - e^{-\frac{1}{\tau_1}(t-u)} \right) dW_u
$$

$$
= \int_0^t \Gamma(u; t) \, dW_u \quad \text{where} \quad \Gamma(u; t) = \sigma \left( e^{-\frac{1}{\tau_2}(t-u)} - e^{-\frac{1}{\tau_1}(t-u)} \right)
$$

(22)

Since $\Gamma(u; t)$ is square integrable, $\mathbb{E} \left[ \int_0^t \Gamma^2(u; t) \, du \right] < \infty$, and for all $u \leq t$, $\Gamma(u; t)$ is $\mathcal{F}_t$ adapted, the conditions of Theorem 4.3.1 [Shreve(2004)] are met and $M_t$ is a martingale. This implies that given $0 < s < t$

$$
\mathbb{E} [M_t | \mathcal{F}_s] = M_s
$$

(23)

$\Gamma(0; 0) = 0$ and $W_0 = 0$ implies that $M_0 = 0$ which combined with the martingale property expressed in equation 23 implies for all $t > 0$ $\mathbb{E} [M_t] = 0$. 
Expected Value and Variance of $\Psi_t$

Theorem 4.3.1 [Shreve(2004)] also implies

$$
\mathbb{E}[\Psi_t] = \mathbb{E}
\left[
\left(\mu - \frac{\sigma^2}{2}\right)(\tau_2 - \tau_1) + \sigma \int_0^t \left(e^{-\frac{1}{\tau_2} (t-u)} - e^{-\frac{1}{\tau_1} (t-u)}\right) dW_u
\right]
= \left(\mu - \frac{\sigma^2}{2}\right)(\tau_2 - \tau_1)
$$

(24)
Expected Value and Variance of $\Psi_t$

Only the Itô integral contributes to the variance of $\Psi_t$

\[
\mathbb{V} [\Psi_t] = \mathbb{V} [M_t]
\]

\[
= \mathbb{E} [(M_t - \mathbb{E} [M_t])^2] = \mathbb{E} [M_t^2]
\]

\[
= \mathbb{E} \left[ \int_0^t \Gamma^2_{(u,t)} \, du \right]
\]

\[
= \mathbb{E} \left[ \sigma^2 \int_0^t \left( e^{-\frac{1}{\tau_2} (t-u)} - e^{-\frac{1}{\tau_1} (t-u)} \right)^2 \, du \right]
\]

\[
= \sigma^2 \int_0^t \left( e^{-\frac{1}{\tau_2} (t-u)} - e^{-\frac{1}{\tau_1} (t-u)} \right)^2 \, du
\]

\[
= \sigma^2 \int_0^t \left( e^{-\frac{2}{\tau_2} (t-u)} - 2e^{-\frac{1}{\tau_1} \left( \frac{1}{\tau_2} + \frac{1}{\tau_1} \right) (t-u)} + e^{-\frac{2}{\tau_1} (t-u)} \right) \, du
\]

\[
= \sigma^2 \left( \left[ \frac{\tau_2}{2} e^{-\frac{2}{\tau_2} (t-u)} \right]_0^t - \left[ \frac{2}{\tau_1 + \frac{1}{\tau_2}} e^{-\left( \frac{1}{\tau_1} + \frac{1}{\tau_2} \right) (t-u)} \right]_0^t + \left[ \frac{\tau_1}{2} e^{-\frac{2}{\tau_1} (t-u)} \right]_0^t \right)
\]

\[
= \sigma^2 \left( \frac{\tau_2}{2} \left( 1 - e^{-\frac{2}{\tau_2} t} \right) - \frac{2}{\tau_1 + \frac{1}{\tau_2}} \left( 1 - e^{-\left( \frac{1}{\tau_1} + \frac{1}{\tau_2} \right) t} \right) + \frac{\tau_1}{2} \left( 1 - e^{-\frac{2}{\tau_1} t} \right) \right)
\]

\[
\text{(25)}
\]
Strategy Analysis: Steady State

Variance of $\Psi_t$ for parameter values $\sigma = 0.1$, $\tau_1 = 5$ and $\tau_2 = 10$

![Graph showing variance of $\Psi_t$ over time](image)
When \( t = 0 \) in equation 25, the variance of \( M_t \) is equal to zero which is expected since \( W_0 = 0 \). Consider the limit of equation 25 as \( t \) goes to infinity.

\[
\lim_{{t \to \infty}} \mathbb{V} [M_t] = \lim_{{t \to \infty}} \left[ \sigma^2 \left( \frac{\tau_2}{2} - \frac{2}{\tau_1 + \frac{1}{\tau_2}} \right) \right] = \sigma^2 \left( \frac{\tau_2}{2} - \frac{2}{\tau_1 + \frac{1}{\tau_2}} \right) = \frac{\sigma^2}{2} \left( \frac{\tau_2 - \tau_1}{\tau_1 + \tau_2} \right)
\]

(26)

Since \( 0 < \tau_1 < \tau_2 \) the steady state variance of \( \Psi_t \) is always greater than zero which is expected as it should only be zero if the two exponentially filtered stochastic process have the same value of \( \tau \). We assume that

\[
\left| \lim_{{t \to \infty}} \mathbb{V} [M_t] - \mathbb{V} [M_{tss}] \right| < \delta
\]

for some sufficiently small \( \delta \).
Distribution of $\Psi_t$

Since $M_t$ is an Itô integral with respect to Brownian motion with a deterministic integrand $\Gamma_{(u,t)}$, by Theorem 4.4.9 [Shreve(2004)] for each $t > 0$, $M_t$ is normally distributed with expected value zero and variance $\int_0^t \Gamma_{(u,t)}^2 \, du$. Thus, $\Psi_t$ from equation 21 is normally distributed with the expected value $u$ and variance $s$ given by equations 24 and 26.

$$\Psi_t \sim \mathcal{N}(u, s^2)$$

$$\sim \mathcal{N} \left( \left( \mu - \frac{\sigma^2}{2} \right) (\tau_2 - \tau_1), \frac{\sigma^2}{2} \left( \frac{(\tau_2 - \tau_1)^2}{\tau_1 + \tau_2} \right) \right) \quad (28)$$

Note that for $t > t_{ss}$ mean and variance are constant and not dependent on $t$, but on the constants $\mu$, $\sigma$, $\tau_1$, and $\tau_2$. 
Strategy Analysis: Steady State

\[ V[\Psi_t] = \frac{\sigma^2}{2} \left( \frac{(\tau_2 - \tau_1)^2}{\tau_1 + \tau_2} \right) \]

Plot of \( V(\Psi_t) \) vs \( \tau_1 \) and \( \tau_2 \)
Density of $\Psi_t$

Given that $\Psi_t$ is normal with mean $u$ and variance $s^2$, $\Psi_t \sim \mathcal{N}(u, s)$, the cumulative distribution function (CDF) $F_{\Psi}$ and probability density function (PDF) $f_{\Psi}$ for $\Psi_t$ are

$$F_{\Psi} = \Phi \left( \frac{\psi - u}{s} \right)$$

$$f_{\Psi} = \frac{d F_{\Psi}}{d \psi} = \frac{1}{s} \phi \left( \frac{\psi - u}{s} \right)$$

where $\Phi$ is the standard normal CDF and $\phi$ is the standard normal PDF. Substitution of $u$ and $s$ from equation 28 into equation 30 gives the full density function for $\Psi_t$. Once the steady state regime is reached the density function has no dependence on $t$.

$$f_{\Psi} = \frac{1}{\sigma (\tau_2 - \tau_1) \sqrt{\frac{\pi}{\tau_1 + \tau_2}}} \exp \left[ - \frac{\left( \psi - \left( \mu - \frac{\sigma^2}{2} \right) (\tau_2 - \tau_1) \right)^2}{\frac{\sigma^2 (\tau_2 - \tau_1)^2}{(\tau_1 + \tau_2)^2}} \right]$$

(31)
Expected Value of \( \varrho_t \)

The position indicator \( \varrho_t \) from equation 5 is a function of the random variable \( \Psi_t \) whose distribution we have from equation 28; thus, we can calculate the expected value of the position indicator using Theorem 3.2.2 part (ix) [Itô(1984)].

\[
E[\varrho_t] = \int_{-\infty}^{\infty} \text{sgn}(\psi) f_{\Psi}(\psi) \, d\psi \\
= \int_{-\infty}^{\infty} \text{sgn}(\psi) \frac{1}{s} \phi \left( \frac{\psi - u}{s} \right) \, d\psi \\
= \int_{-\infty}^{0} (-1) \frac{1}{s} \phi \left( \frac{\psi - u}{s} \right) \, d\psi + \int_{0}^{\infty} (1) \frac{1}{s} \phi \left( \frac{\psi - u}{s} \right) \, d\psi \\
= \lim_{a \to \infty} \left[ \Phi \left( \frac{\psi - u}{s} \right) \right]_{a}^{0} - \lim_{b \to -\infty} \left[ \Phi \left( \frac{\psi - u}{s} \right) \right]_{b}^{0} \\
= \left[ 1 - \Phi \left( \frac{0 - u}{s} \right) \right] - \left[ \Phi \left( \frac{0 - u}{s} \right) - \Phi \left( \frac{0 - u}{s} \right) \right] \\

Expected Value of $\varrho_t$ Continued

\[
\mathbb{E} [\varrho_t] = 1 - 2 \Phi \left( -\frac{u}{s} \right) \\
= 1 - 2 \Phi \left( -\frac{\left( \mu - \frac{\sigma^2}{2} \right) (\tau_2 - \tau_1)}{\frac{\sigma^2}{2} \left( \frac{(\tau_2 - \tau_1)^2}{\tau_1 + \tau_2} \right)} \right) \\
= 1 - 2 \Phi \left[ -\frac{2}{\sigma^2} \left( \mu - \frac{\sigma^2}{2} \right) \frac{\tau_1 + \tau_2}{\tau_2 - \tau_1} \right]
\]

(32)
**Log Return Expected Value**

We have now assembled the results needed to calculate the Luxor steady state expected log return $r_t$ as given in definition 16.

\[
\mathbb{E} [r_t] = \mathbb{E} \left[ \int_0^t \varrho_u d (\ln S_u) \right] \\
= \mathbb{E} \left[ \int_0^t \varrho_u \left( \mu - \frac{\sigma^2}{2} \right) du + \sigma dW_u \right] \\
= \mathbb{E} \left[ \left( \mu - \frac{\sigma^2}{2} \right) \int_0^t \varrho_u du + \sigma \int_0^t \varrho_u dW_u \right] \\
= \left( \mu - \frac{\sigma^2}{2} \right) \mathbb{E} \left[ \int_0^t \varrho_u du \right] + \sigma \mathbb{E} \left[ \int_0^t \varrho_u dW_u \right] \\
\]  
(33)
Log Return

The position indicator $\varrho_t$ by definition is a random sequence of the elements in the set $\{-1, 0, 1\}$ which implies

$$\int_0^t \varrho_u^2 \, du \leq t < \infty$$

thus $\varrho_t^2$ is square integrable on $[0, t]$. In addition, $\varrho_t$ is $\mathcal{F}_t$ adapted; therefore, the Itô integral inside the second expected value in equation 33 mets the conditions of Theorem 4.3.1 [Shreve(2004)] and the integral is a martingale with an initial and expected value of zero.

Define the function $g$ such that

$$\xi = g(\rho) = \int_0^t \rho \, du$$

and define the random variable $\Xi_t$.

$$\Xi_t = g(\varrho_t) = \int_0^t \varrho_u \, du$$
Log Return

Since the last term in equation 33 is zero, the expected value of the log return is given by the first term.

$$
E[r_t] = \left( \mu - \frac{\sigma^2}{2} \right) E \left[ \int_0^t \varrho u \, du \right]
$$

$$
= \left( \mu - \frac{\sigma^2}{2} \right) E[\Xi_t]
$$

$$
= \left( \mu - \frac{\sigma^2}{2} \right) \int_{-\infty}^{\infty} \xi f_{\Xi_t}(\xi) \, d\xi
$$

Substitute $\xi = g(\rho)$ and $f_{\Xi_t}(\xi) \, d\xi = f_{\varrho_\lambda}(\rho) \, d\rho$

via Theorem 3.2.2 part (ix) in [Ito(1984)]

$$
= \left( \mu - \frac{\sigma^2}{2} \right) \int_{-\infty}^{\infty} g(\rho) f_{\varrho_\lambda}(\rho) \, d\rho
$$

$$
= \left( \mu - \frac{\sigma^2}{2} \right) \int_{-\infty}^{\infty} \left( \int_0^t \rho \, du \right) f_{\varrho_\lambda}(\rho) \, d\rho
$$

(37)

(38)
**Strategy Analysis: Log Return**

**Log Return**

Change the order of integration via Fubini’s theorem

\[
\begin{align*}
&= \left( \mu - \frac{\sigma^2}{2} \right) \int_0^t \left( \int_{-\infty}^\infty \rho f_{\epsilon_t}(\rho) \, d\rho \right) \, du \\
&= \left( \mu - \frac{\sigma^2}{2} \right) \int_0^t \mathbb{E}[\varrho_u] \, du
\end{align*}
\]

(39)

Inserting the expected value of \( \varrho \) from equation 32 into equation 39 and solving the integral gives the expected log return.

\[
\mathbb{E}[r_t] = \left( \mu - \frac{\sigma^2}{2} \right) \int_0^t \left[ 1 - 2\Phi \left( -\frac{2}{\sigma^2} \left( \mu - \frac{\sigma^2}{2} \right) \frac{\tau_1 + \tau_2}{\tau_2 - \tau_1} \right) \right] \, du \\
= \left( \mu - \frac{\sigma^2}{2} \right) \left( 1 - 2\Phi \left[ -\frac{2}{\sigma^2} \left( \mu - \frac{\sigma^2}{2} \right) \frac{\tau_1 + \tau_2}{\tau_2 - \tau_1} \right] \right) \, t
\]

(40)
Proposition:
Given the assumptions and model in sections 5 and 6, the expected steady state log return of the Luxor strategy is greater than or equal to zero.

\[ E[r_t] \geq 0 \] (41)

Proof:
We proceed by determining the sign of each term in equation 40. \( t \) and \( \sigma^2 \) are positive by definition, and \( \frac{\tau_1 + \tau_2}{\tau_2 - \tau_1} > 0 \) since \( 0 < \tau_1 < \tau_2 \). Let

\[ C = -\frac{2}{\sigma^2} \left( \mu - \frac{\sigma^2}{2} \right) \frac{\tau_1 + \tau_2}{\tau_2 - \tau_1} \] (42)

Thus the order relation between \( \mu \) and \( \frac{\sigma^2}{2} \) controls the sign of \( C \).
Proof (Continued):

Table 1 gives the value of $\mu$ relative to $\frac{\sigma^2}{2}$ in the left column, the sign of each term that can take on a negative value in the center columns and the resulting expected log return $E[r_t]$ in the right hand column.

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>C</th>
<th>$\Phi$</th>
<th>$1 - 2\Phi$</th>
<th>$\left(\mu - \frac{\sigma^2}{2}\right)$</th>
<th>$E[r_t]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$&gt; \frac{\sigma^2}{2}$</td>
<td>-</td>
<td>$&lt;\frac{1}{2}$</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>$= \frac{\sigma^2}{2}$</td>
<td>0</td>
<td>$=\frac{1}{2}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$&lt; \frac{\sigma^2}{2}$</td>
<td>+</td>
<td>$&gt;\frac{1}{2}$</td>
<td>-</td>
<td>-</td>
<td>+</td>
</tr>
</tbody>
</table>

Table 1: Expected Log Return Relative to $\mu$ and $\sigma$

Comparing the far left hand column with the far right hand column in table 1, we see that for every $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}_{\geq 0}$, and $0 < \tau_1 < \tau_2$, the expected log return $r_t$ is greater than or equal to zero.
\[ \frac{d \mathbb{E}[r_t]}{dt} = \left( \mu - \frac{\sigma^2}{2} \right) \left( 1 - 2\Phi \left[ -\frac{2}{\sigma^2} \left( \mu - \frac{\sigma^2}{2} \right) \frac{\tau_1 + \tau_2}{\tau_2 - \tau_1} \right] \right) \]

\[ \sigma = 0.1, \tau_1 = 3, \tau_2 = 20, t = 1 \]
Strategy Analysis: Log Return

\[
\frac{d \mathbb{E}[r_t]}{dt} = \left( \mu - \frac{\sigma^2}{2} \right) \left( 1 - 2 \Phi \left[ -\frac{2}{\sigma^2} \left( \mu - \frac{\sigma^2}{2} \right) \frac{\tau_1 + \tau_2}{\tau_2 - \tau_1} \right] \right) \quad \sigma = 0.3, \tau_1 = 3, \tau_2 = 20, t = 1
\]
Summary
**Summary**

**Results**

Given the assumptions and model in sections 5 and 6, we

- Derived a closed form solution for the expected log returns of Luxor
- Demonstrated Luxor has a non negative expected value
- When $\mu > \frac{\sigma^2}{2}$ high volatility causes under performance compared to buy and hold

**Issues**

Issues we are currently addressing

- Simple SDE model does not capture features of stock prices that may be advantageous to strategies like Luxor
- More accurate price SDE models
- More realistic transaction and slippage models
Fischer Black and Myron Scholes. 
**The pricing of options and corporate liabilities.**

B. Efron.  
**Bootstrap methods: Another look at the jackknife.**  

Kiyosi Itô. 
**Introduction to Probability Theory.**  

Urban Jaekle and Emilio Tomasini. 
**Trading Systems - A New Approach to System Development and Portfolio Optimization.**  

Robert C. Merton. 
**Theory of rational option pricing.**  

