QUANTITATIVE ANALYSIS OF DUAL MOVING AVERAGE INDICATORS IN AUTOMATED TRADING SYSTEMS

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OUTLINE

INTRODUCTION

Dual Moving Average Technical Trading Strategy [Jaekle and Tomasini(2009)]

Long position - fast MA above slow MA

Short position - fast MA below slow MA

INTRODUCTION

Luxor dual moving average trading strategy

Luxor historically analyzed via inferential statistics

- Discrete time setting
- Historical data [Jaekle and Tomasini(2009)]
- Bootstrapped resampled [Efron(1979)]) [William Brock(1992)]

to determine

- Expected returns
- Draw down
- Select strategy parameters etc.

Our goal

Assume price dynamics are controlled by a stochastic differential equation (SDE) and derive the Luxor closed form expected log returns in a continuous time setting

- Inspired by work in options pricing [Black and Scholes(1973), Merton(1973)]
- Gain analytical tractability by using the natural logarithm of the price
- Build frame work for analyzing trading strategies

ASSUMPTIONS

ASSUMPTIONS

To gain analytical tractability we assume a frictionless market

1. Difference between the bid and ask price is infinitesimal
2. Transaction costs are zero
3. Transaction execution is instantaneous
4. Buy (sell) transactions fully execute at the current ask (bid) price
5. Market contains a single risky asset with price S_t

CONTINUOUS TIME MODEL

CONTINUOUS TIME MODEL

Price Dynamics

- Let $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ be a complete filtered probability space
- Dynamics of S_t under $\mathbb{P}$ is controlled by the SDE

$$dS_t = \mu S_t dt + \sigma S_t dW_t \qquad \mu,\ \sigma \in \mathbb{R} \text{ constant}$$

- W_t is one dimensional Brownian motion whose history up to time t contained in the filtration $\{\mathcal{F}_t\}_{t\geq 0}$
- For each $\omega \in \Omega$ there exits a function

$$W_t(\omega) : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+ \quad \text{(the samples)}$$

- $W_0(\omega) = 0$, disjoint increments of $W_t(\omega)$ are independent, increments $W_{\Delta t}(\omega)$ are normally distributed with mean zero and variance Δt, and $W_t(\omega)$ is continuous on $[0, t]$.

CONTINUOUS TIME MODEL

SDE Solution

Let

$$f(s) = \ln s, \quad f'(s) = \frac{1}{s}, \quad f''(s) = -\frac{1}{s^2}$$
$$(dt)^2 = 0, \quad (dt)(dW_t) = 0, \quad (dW_t)^2 = dt \quad ([W, W]_t = t)$$

Apply Itô's formula [Shreve(2004)] to $d \ln S_t$ where $dS_t = \mu S_t dt + \sigma S_t dW_t$

$$\begin{aligned}
df(S_t) &= f'(S_t)\, dS_t + \frac{1}{2} f''(S_t)\, (dS_t)^2 \\
d \ln S_t &= \frac{1}{S_t} dS_t + \frac{1}{2}\left(-\frac{1}{S_t^2}\right)(dS_t)^2 \\
&= \frac{1}{S_t}\left(\mu S_t dt + \sigma S_t dW_t\right) + \frac{1}{2}\left(-\frac{1}{S_t^2}\right)\left(\sigma^2 S_t^2 dt\right) \\
&= \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dW_t \qquad (1)
\end{aligned}$$

CONTINUOUS TIME MODEL

SDE Solution

Integrate equation 1 from 0 to t and apply exponential function to both sides

$$\int_0^t d\ln S_t = \int_0^t \left(\mu - \frac{\sigma^2}{2}\right) du + \int_0^t \sigma dW_u$$
$$\ln S_t = \ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t$$
$$S_t = S_0 \, e^{(\mu - \frac{\sigma^2}{2})\, t + \sigma \, W_t} \qquad (2)$$

The solution S_t is known as geometric Brownian motion.

CONTINUOUS TIME MODEL

Model

Dividing both sides of the SDE by S_t, it can be rewritten as

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \tag{3}$$

In essence, the instantaneous return is a constant term plus a volatility term

- If we knew that either $\mu >> \frac{\sigma^2}{2}$ or $\mu << \frac{\sigma^2}{2}$ and the SDE model is accurate, investing would be easy; thus, we assume $|\mu - \frac{\sigma^2}{2}|$ is not large and μ is not easy to determine for a given asset over a short time frame

CONTINUOUS TIME MODEL

Exponentially Filtered Stochastic Process

An exponentially filtered stochastic process $\Upsilon_t[X_t;\tau]$ where $\tau > 0$ is a convolution of the exponential kernel function $\frac{1}{\tau}e^{-\frac{1}{\tau}t}$ with the stochastic process X_t.

$$\Upsilon_t[X_t;\tau] = \int_0^t \frac{1}{\tau} e^{-\frac{1}{\tau}(t-u)} X_u du \tag{4}$$

Note:

- $\Upsilon_t[X_t;\tau]$ is a path dependent time series operator
- $\Upsilon_t[X_t;\tau]$ is $\{\mathcal{F}_t\}_{t\geq 0}$ measurable - can calculate $\Upsilon_t[X_t;\tau]$ given filtration at time t
- Lower limit is usually defined as $-\infty$; however, we work with processes on $[0, T]$ and assume a steady state is reached in finite time at $t > t_{ss}$

STRATEGY DEFINITION

STRATEGY DEFINITION

Strategy Components [Peterson(2015)]

- Filters - select instruments
- Indicators - quantitative values derived from market data
- Signals - respond to interactions between filters, market data, and indicators
- Rules - make path dependent actionable decisions when signals fire

Formulate Luxor as a set of equations following this model

STRATEGY DEFINITION

Difference Indicator

The difference indicator $\Psi_t : \mathbb{R} \to \mathbb{R}$ is the difference of two exponential filters applied to a stochastic process X_t where $0 < \tau_1 < \tau_2$.

$$\Psi_t[X_t; \tau_1, \tau_2] = \Upsilon_t[X_t; \tau_1] - \Upsilon_t[X_t; \tau_2]$$

- Use natural logarithm of the asset price $X_t = \ln S_t$ to gain analytical tractability
- $\Psi_t[X_t; \tau_1, \tau_2]$ known as moving average convergence divergence (MACD) oscillator in technical analysis.

STRATEGY DEFINITION

Position Indicator

The position indicator $\varrho_t : \mathbb{R} \to \{-1, 0, 1\}$ is the composition of the signum function with the difference indicator and indicates potential long or short position opportunities.

$$\varrho_t = (\operatorname{sgn} \circ \Psi)_t = \operatorname{sgn}(\Psi_t) = \begin{cases} 1 & \Psi_t \geq 0 & \text{long} \\ 0 & \Psi_t = 0 & \text{previous} \\ -1 & \Psi_t < 0 & \text{short} \end{cases} \tag{5}$$

STRATEGY DEFINITION

Position Entry/Exit Signals

Position indicator transitions

- $(-1 \rightarrow +1)$ or $(-1 \rightarrow 0 \rightarrow +1)$ exit any short position and enter a long position
- $(1 \rightarrow -1)$ or $(1 \rightarrow 0 \rightarrow -1)$ exit any long position and enter short position

Initial entry/final exit occur on first position indicator entry/exit signal after steady state is reached at $t > t_{ss}$

STRATEGY DEFINITION

Rules

Always act on position indicator signals (identity function)

- Gain analytical tractability
- Efficient implementation uses rules to define
    - When and how to enter or exit the market
    - Determine position size
    - Manage risk

EVALUATION CRITERIA

EVALUATION CRITERIA

Luxor Log Return

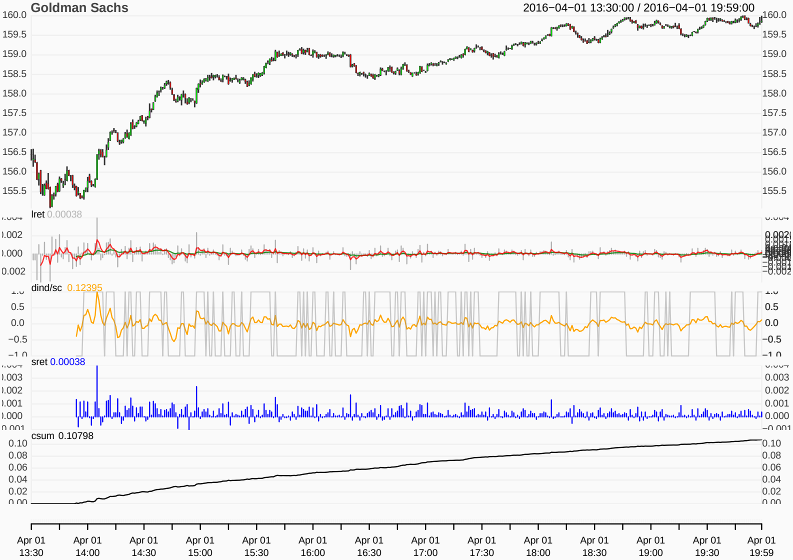
The log return r_t of the Luxor strategy on $[0, t]$ is given by the integral of the position indicator ϱ with respect to the change of the logarithm of the asset price.

$$r_t = \int_0^t \varrho_u d \ln S_u \tag{6}$$

Assumes equal size positions in the asset

Example

EXAMPLE


Goldman Sachs
2016−04−01 13:30:00 / 2016−04−01 19:59:00
160.0
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lret 0.00038
0.002
0.000
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dind/sc 0.12395
1.0
0.5
0.0
−0.5
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sret 0.00038
0.003
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csum 0.10798
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STRATEGY ANALYSIS

STRATEGY ANALYSIS

Exponentially Filtered Logarithm of Asset Price

Construct the exponentially filtered logarithm of the asset price $\Upsilon_t[\ln S_t, \tau]$ by substituting the logarithm of the SDE solution S_t from equation 2 for X_t in the filter definition in equation 4.

$$
\begin{aligned}
\Upsilon_t[\ln S_t, \tau] &= \int_0^t \frac{1}{\tau} e^{-\frac{1}{\tau}(t-u)} \left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2} \right) u + \sigma W_u \right] du \\
&= \frac{e^{-\frac{1}{\tau}t}}{\tau} \left[\ln S_0 \int_0^t e^{\frac{1}{\tau}u} du + \left(\mu - \frac{\sigma^2}{2} \right) \int_0^t u\, e^{\frac{1}{\tau}u} du + \sigma \int_0^t e^{\frac{1}{\tau}u} W_u du \right] \qquad (7)
\end{aligned}
$$

First two integrals in $\Upsilon_t[\ln S_t, \tau]$ are deterministic Riemann integrals wrt time and have explicit solutions. The last integral is a stochastic Riemann integral of Brownian motion wrt time.

STRATEGY ANALYSIS

Exponentially Filtered Logarithm of Asset Price

First integral in $\Upsilon_t[\ln S_t, \tau]$, equation is solved using the fundamental theorem of calculus.

$$\begin{aligned} \int_0^t e^{\frac{1}{\tau}u} du &= \tau \left[e^{\frac{1}{\tau}u} \right]_0^t \\ &= \tau \left(e^{\frac{1}{\tau}t} - 1 \right) \end{aligned} \tag{8}$$

The second integral requires in addition to the fundamental theorem of calculus integration by parts where $\int xdy = xy - \int ydx$. Let $x = u$ which implies $dx = du$, and let

$$\begin{aligned} dy &= e^{\frac{1}{\tau}u} du \\ y &= \int e^{\frac{1}{\tau}u} du \\ &= \tau e^{\frac{1}{\tau}u} \end{aligned} \tag{9}$$

STRATEGY ANALYSIS

Exponentially Filtered Logarithm of Asset Price

Using the results from equation 9 to complete the integration by parts solves the second integral in $\Upsilon_t[\ln S_t, \tau]$ from equation 7.

$$\begin{aligned}\int_0^t u\, e^{\frac{1}{\tau}u} du &= \tau\left[ue^{\frac{1}{\tau}u}\right]_0^t - \int_0^t \tau e^{\frac{1}{\tau}u} du \\ &= \tau t e^{\frac{1}{\tau}t} - \tau^2\left[e^{\frac{1}{\tau}u}\right]_0^t \\ &= \tau\left[te^{\frac{1}{\tau}t} - \tau\left(e^{\frac{1}{\tau}t} - 1\right)\right] \qquad (10)\end{aligned}$$

STRATEGY ANALYSIS

Exponentially Filtered Logarithm of Asset Price

Inserting the results of equations 8 and 10 into $\Upsilon_t[\ln S_t, \tau]$, equation 7, results in an equation with two deterministic terms and a stochastic term which is a Riemann integral of scaled Brownian motion wrt time.

$$\begin{aligned}\Upsilon_t[\ln S_t, \tau] &= \frac{e^{-\frac{1}{\tau}t}}{\tau}\left[\tau \ln S_0 \left(e^{\frac{1}{\tau}t} - 1\right)\right. \\ &\qquad \left. + \tau\left(\mu - \frac{\sigma^2}{2}\right)\left[te^{\frac{1}{\tau}t} - \tau\left(e^{\frac{1}{\tau}t} - 1\right)\right] + \sigma \int_0^t e^{\frac{1}{\tau}u} W_u du\right] \\ &= \ln S_0 \left(1 - e^{-\frac{1}{\tau}t}\right) \\ &\qquad + \left(\mu - \frac{\sigma^2}{2}\right)\left[t - \tau\left(1 - e^{-\frac{1}{\tau}t}\right)\right] + \frac{\sigma e^{-\frac{1}{\tau}t}}{\tau}\int_0^t e^{\frac{1}{\tau}u} W_u du \end{aligned} \tag{11}$$

STRATEGY ANALYSIS

Difference Indicator

Let $0 < \tau_1 < \tau_2$, the difference indicator $\Psi_t[\ln S_t, \tau_1, \tau_2]$ is formed from the difference of two versions of Υ with different values of τ applied to $\ln S_t$.

$$
\begin{aligned}
\Psi_t[\ln S_t, \tau_1, \tau_2] &= \Upsilon_t[\ln S_t, \tau_1] - \Upsilon_t[\ln S_t, \tau_2] \\
&= \ln S_0 \left(1 - e^{-\frac{1}{\tau_1}t}\right) + \left(\mu - \frac{\sigma^2}{2}\right)\left[t - \tau_1\left(1 - e^{-\frac{1}{\tau_1}t}\right)\right] + \frac{\sigma e^{-\frac{1}{\tau_1}t}}{\tau_1}\int_0^t e^{\frac{1}{\tau_1}u} W_u du \\
&\quad - \ln S_0 \left(1 - e^{-\frac{1}{\tau_2}t}\right) - \left(\mu - \frac{\sigma^2}{2}\right)\left[t - \tau_2\left(1 - e^{-\frac{1}{\tau_2}t}\right)\right] - \frac{\sigma e^{-\frac{1}{\tau_2}t}}{\tau_2}\int_0^t e^{\frac{1}{\tau_2}u} W_u du \\
&= \ln S_0 \left(e^{-\frac{1}{\tau_2}t} - e^{-\frac{1}{\tau_1}t}\right) \\
&\qquad + \left(\mu - \frac{\sigma^2}{2}\right)\left[\tau_2\left(1 - e^{-\frac{1}{\tau_2}t}\right) - \tau_1\left(1 - e^{-\frac{1}{\tau_1}t}\right)\right] \\
&\qquad\quad + \sigma \int_0^t \left(\frac{1}{\tau_1} e^{-\frac{1}{\tau_1}(t-u)} - \frac{1}{\tau_2} e^{-\frac{1}{\tau_2}(t-u)}\right) W_u \, du
\end{aligned}
\tag{12}
$$

From this point forward $\Psi_t = \Psi_t[\ln S_t, \tau_1, \tau_2]$.

Convert to Itô Integral

When f_t is deterministic, the Itô product rule [Oksendal(2000)], reduces to

$$\begin{aligned} d(f_t W_t) &= (df_t) W_t + f_t dW_t + (df_t)(dW_t) \\ &= f_t' dt W_t + f_t dW_t \end{aligned}$$

Rearranging terms and integrating both sides from 0 to t gives an integration by parts formula that converts a Riemann integral of Brownian motion wrt time to a scaled Brownian motion term and an Itô integral

$$\begin{aligned} f_t' W_t dt &= d(f_t W_t) - f_t dW_t \\ \int_0^t f_u' W_u du &= \int_0^t d(f_u W_u) - \int_0^t f_u dW_u \\ \int_0^t f_u' W_u du &= [f_u W_u]_0^t - \int_0^t f_u dW_u \qquad (13) \end{aligned}$$

STRATEGY ANALYSIS: DIFFERENCE INDICATOR

Convert to Itô Integral

Let

$$\begin{aligned}
\frac{d\,f_u}{d\,u} &= \sigma\left(\frac{1}{\tau_1}e^{-\frac{1}{\tau_1}(t-u)} - \frac{1}{\tau_2}e^{-\frac{1}{\tau_2}(t-u)}\right)\\
\int df_u &= \sigma\int\left(\frac{1}{\tau_1}e^{-\frac{1}{\tau_1}(t-u)} - \frac{1}{\tau_2}e^{-\frac{1}{\tau_2}(t-u)}\right)du\\
f_t &= \sigma\left(e^{-\frac{1}{\tau_1}(t-u)} - e^{-\frac{1}{\tau_2}(t-u)}\right) \qquad (14)
\end{aligned}$$

Rearrange terms as in equation 13

$$\begin{aligned}
&\sigma\int_0^t\left(\frac{1}{\tau_1}e^{-\frac{1}{\tau_1}(t-u)} - \frac{1}{\tau_2}e^{-\frac{1}{\tau_2}(t-u)}\right)W_u\,du\\
&\quad= \sigma\cancelto{0}{\left[\left(e^{-\frac{1}{\tau_1}(t-u)} - e^{-\frac{1}{\tau_2}(t-u)}\right)W_u\right]_0^t} - \sigma\int_0^t\left(e^{-\frac{1}{\tau_1}(t-u)} - e^{-\frac{1}{\tau_2}(t-u)}\right)\,dW_u\\
&\quad= \sigma\int_0^t\left(e^{-\frac{1}{\tau_2}(t-u)} - e^{-\frac{1}{\tau_1}(t-u)}\right)\,dW_u \qquad (15)
\end{aligned}$$

STRATEGY ANALYSIS: DIFFERENCE INDICATOR

Itô Process

Substituting the results of the integration by parts into equation 12 converts Ψ_t to an Itô process

$$\begin{aligned}\Psi_t = {} & \ln S_0 \left(e^{-\frac{1}{\tau_2}t} - e^{-\frac{1}{\tau_1}t}\right) \\ & + \left(\mu - \frac{\sigma^2}{2}\right)\left[\tau_2\left(1 - e^{-\frac{1}{\tau_2}t}\right) - \tau_1\left(1 - e^{-\frac{1}{\tau_1}t}\right)\right] \\ & + \sigma \int_0^t \left(e^{-\frac{1}{\tau_2}(t-u)} - e^{-\frac{1}{\tau_1}(t-u)}\right) \, dW_u \end{aligned} \tag{16}$$

STRATEGY ANALYSIS: DIFFERENCE INDICATOR

Transient Behavior

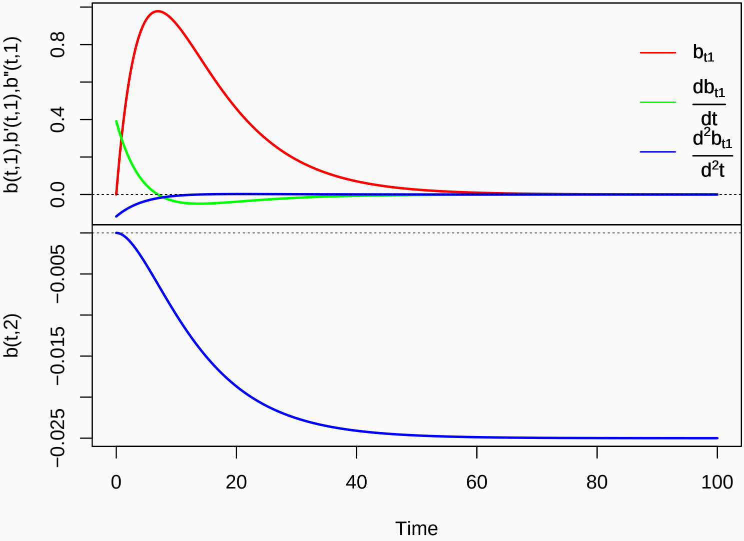
Define the first two terms of Ψ_t equation 16 as

$$b_{t,1} = \ln S_0 \left[e^{-\frac{1}{\tau_2} t} - e^{-\frac{1}{\tau_1} t} \right] \tag{17}$$

$$b_{t,2} = \left(\mu - \frac{\sigma^2}{2} \right) \left[\tau_2 \left(1 - e^{-\frac{1}{\tau_2} t} \right) - \tau_1 \left(1 - e^{-\frac{1}{\tau_1} t} \right) \right] \tag{18}$$

STRATEGY ANALYSIS: DIFFERENCE INDICATOR

Plot of $b_{t,1}$ **and** $b_{t,2}$ **with** $\mu = 0$**,** $\sigma = 0.1$**,** $\tau_1 = 5$**, and** $\tau_2 = 10$



STRATEGY ANALYSIS: DIFFERENCE INDICATOR

Steady State Assumptions

We assume that for $S_0 > 1$ and $0 < \tau_1 < \tau_2$ there exists a t_{ss} such that for all $t > t_{ss}$ the transient behavior of the deterministic terms $b_{t,1}$ and $b_{t,2}$ is insignificant to the analysis and can be ignored.

$$\begin{aligned}
&\lim_{t\to\infty} [b_{t,1} + b_{t,2}] \\
&\quad = \lim_{t\to\infty} \left[\ln S_0 \left(e^{-\frac{1}{\tau_2}t} - e^{-\frac{1}{\tau_1}t}\right) + \left(\mu - \frac{\sigma^2}{2}\right)\left[\tau_2\left(1 - e^{-\frac{1}{\tau_2}t}\right) - \tau_1\left(1 - e^{-\frac{1}{\tau_1}t}\right)\right]\right] \\
&\quad = \left(\mu - \frac{\sigma^2}{2}\right)(\tau_2 - \tau_1) \qquad (19)
\end{aligned}$$

We assume that

$$\left|\lim_{t\to\infty} [b_{t,1} + b_{t,2}] - [b_{t,1} + b_{t,2}]_{t=t_{ss}}\right| < \delta \qquad (20)$$

for some sufficiently small δ.

STRATEGY ANALYSIS: DIFFERENCE INDICATOR

Steady State Difference Indicator

$$\Psi_t = \left(\mu - \frac{\sigma^2}{2}\right)(\tau_2 - \tau_1) + \sigma \int_0^t \left(e^{-\frac{1}{\tau_2}(t-u)} - e^{-\frac{1}{\tau_1}(t-u)}\right) \, dW_u \tag{21}$$

STRATEGY ANALYSIS: STEADY STATE

Expected Value and Variance of Ψ_t

Let M_t represent the Itô integral contained in the definition of Ψ_t in equation 21.

$$\begin{aligned} M_t &= \sigma \int_0^t \left(e^{-\frac{1}{\tau_2}(t-u)} - e^{-\frac{1}{\tau_1}(t-u)} \right) \, dW_u \\ &= \int_0^t \Gamma_{(u;t)} \, dW_u \quad \text{where} \quad \Gamma_{(u;t)} = \sigma \left(e^{-\frac{1}{\tau_2}(t-u)} - e^{-\frac{1}{\tau_1}(t-u)} \right) \end{aligned} \tag{22}$$

Since $\Gamma^2_{(u;t)}$ is square integrable, $\mathbb{E}\left[\int_0^t \Gamma^2_{(u;t)} \, du\right] < \infty$, and for all $u \leq t$, $\Gamma_{(u;t)}$ is $\mathcal{F}_t$ adapted, the conditions of Theorem 4.3.1 [Shreve(2004)] are met and M_t is a martingale. This implies that given $0 < s < t$

$$\mathbb{E}\left[M_t | \mathcal{F}_s\right] = M_s \tag{23}$$

$\Gamma_{(0;0)=0}$ and $W_0 = 0$ implies that $M_0 = 0$ which combined with the martingale property expressed in equation 23 implies for all $t > 0$ $\mathbb{E}\left[M_t\right] = 0$.

Expected Value and Variance of Ψ_t

Theorem 4.3.1 [Shreve(2004)] also implies

$$\begin{aligned}\mathbb{E}\left[\Psi_t\right] &= \mathbb{E}\left[\left(\mu - \frac{\sigma^2}{2}\right)(\tau_2 - \tau_1) + \sigma \int_0^t \left(e^{-\frac{1}{\tau_2}(t-u)} - e^{-\frac{1}{\tau_1}(t-u)}\right)\, dW_u\right] \\ &= \left(\mu - \frac{\sigma^2}{2}\right)(\tau_2 - \tau_1) \qquad (24)\end{aligned}$$

STRATEGY ANALYSIS: STEADY STATE

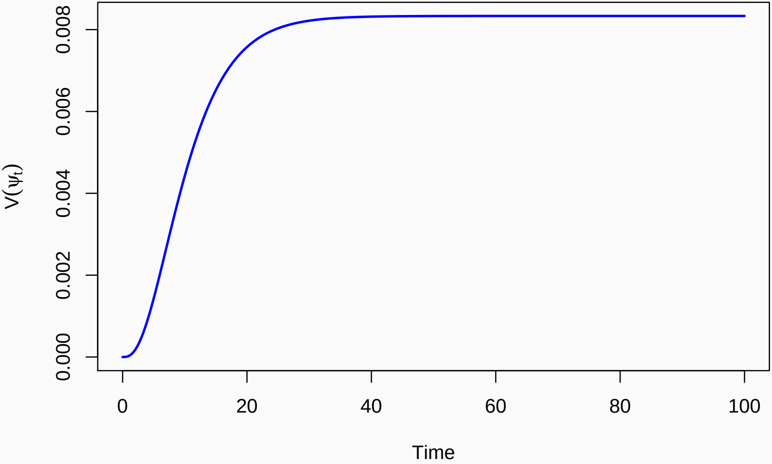
Expected Value and Variance of Ψ_t

Only the Itô integral contributes to the variance of Ψ_t

$$
\begin{aligned}
\mathbb{V}[\ \Psi_t] &= \mathbb{V}[M_t] \\
&= \mathbb{E}\left[(M_t - \mathbb{E}[M_t])^2\right] = \mathbb{E}\left[M_t^2\right] \\
&= \mathbb{E}\left[\int_0^t \Gamma^2_{(u;t)}\,du\right] \\
&= \mathbb{E}\left[\sigma^2 \int_0^t \left(e^{-\frac{1}{\tau_2}(t-u)} - e^{-\frac{1}{\tau_1}(t-u)}\right)^2\ du\right] \\
&= \sigma^2 \int_0^t \left(e^{-\frac{1}{\tau_2}(t-u)} - e^{-\frac{1}{\tau_1}(t-u)}\right)^2\ du \\
&= \sigma^2 \int_0^t \left(e^{-\frac{2}{\tau_2}(t-u)} - 2e^{-\left(\frac{1}{\tau_1}+\frac{1}{\tau_2}\right)(t-u)} + e^{-\frac{2}{\tau_1}(t-u)}\right)\ du \\
&= \sigma^2 \left(\left[\frac{\tau_2}{2} e^{-\frac{2}{\tau_2}(t-u)}\right]_0^t - \left[\frac{2}{\frac{1}{\tau_1}+\frac{1}{\tau_2}} e^{-\left(\frac{1}{\tau_1}+\frac{1}{\tau_2}\right)(t-u)}\right]_0^t + \left[\frac{\tau_1}{2} e^{-\frac{2}{\tau_1}(t-u)}\right]_0^t\right) \\
&= \sigma^2 \left(\frac{\tau_2}{2}\left(1 - e^{-\frac{2}{\tau_2}t}\right) - \frac{2}{\frac{1}{\tau_1}+\frac{1}{\tau_2}}\left(1 - e^{-\left(\frac{1}{\tau_1}+\frac{1}{\tau_2}\right)t}\right) + \frac{\tau_1}{2}\left(1 - e^{-\frac{2}{\tau_1}t}\right)\right) \qquad (25)
\end{aligned}
$$

STRATEGY ANALYSIS: STEADY STATE

Variance of Ψ_t for parameter values $\sigma = 0.1$, $\tau_1 = 5$ and $\tau_2 = 10$



STRATEGY ANALYSIS: STEADY STATE

When $t = 0$ in equation 25, the variance of M_t is equal to zero which is expected since $W_0 = 0$. Consider the limit of equation 25 as t goes to infinity.

$$\begin{aligned}
&\lim_{t\to\infty} \mathbb{V}\left[M_t\right] \\
&\quad = \lim_{t\to\infty} \left[\sigma^2 \left(\frac{\tau_2}{2}\left(1 - e^{-\frac{2}{\tau_2}t}\right) - \frac{2}{\frac{1}{\tau_1} + \frac{1}{\tau_2}}\left(1 - e^{-\left(\frac{1}{\tau_1}+\frac{1}{\tau_2}\right)t}\right) + \frac{\tau_1}{2}\left(1 - e^{-\frac{2}{\tau_1}t}\right)\right)\right] \\
&\quad = \sigma^2 \left(\frac{\tau_2}{2} - \frac{2}{\frac{1}{\tau_1} + \frac{1}{\tau_2}} + \frac{\tau_1}{2}\right) \\
&\quad = \frac{\sigma^2}{2}\left(\frac{\left(\tau_2 - \tau_1\right)^2}{\tau_1 + \tau_2}\right)
\end{aligned} \tag{26}$$

Since $0 < \tau_1 < \tau_2$ the steady state variance of Ψ_t is always greater than zero which is expected as it should only be zero if the two exponentially filtered stochastic process have the same value of τ. We assume that

$$\left| \lim_{t\to\infty} \mathbb{V}\left[M_t\right] - \mathbb{V}\left[M_t\right]\Big|_{t=t_{ss}} \right| < \delta \tag{27}$$

for some sufficiently small δ.

STRATEGY ANALYSIS: STEADY STATE

Distribution of Ψ_t

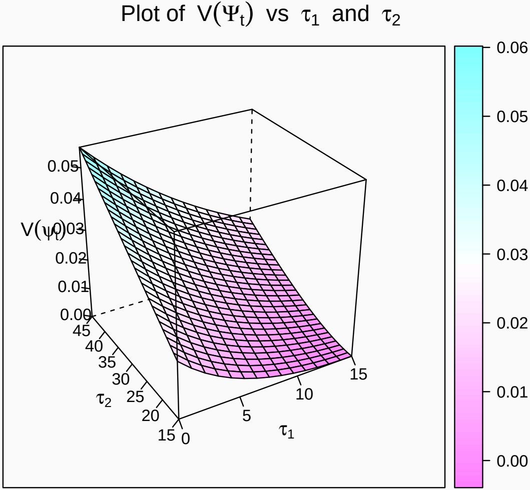
Since M_t is an Itô integral with respect to Brownian motion with a deterministic integrand $\Gamma_{(u;t)}$, by Theorem 4.4.9 [Shreve(2004)] for each $t > 0$, M_t is normally distributed with expected value zero and variance $\int_0^t \Gamma^2_{(u;t)}\, du$. Thus, Ψ_t from equation 21 is normally distributed with the expected value u and variance s given by equations 24 and 26.

$$\begin{aligned}\Psi_t &\sim \mathcal{N}\left(u, s^2\right) \\ &\sim \mathcal{N}\left(\left(\mu - \frac{\sigma^2}{2}\right)(\tau_2 - \tau_1), \frac{\sigma^2}{2}\left(\frac{(\tau_2 - \tau_1)^2}{\tau_1 + \tau_2}\right)\right) \end{aligned} \tag{28}$$

Note that for $t > t_{ss}$ mean and variance are constant and not dependent on t, but on the constants μ, σ, τ_1, and τ_2.

STRATEGY ANALYSIS: STEADY STATE

$$\mathbb{V}\left[\Psi_t\right] = \frac{\sigma^2}{2}\left(\frac{(\tau_2 - \tau_1)^2}{\tau_1 + \tau_2}\right)$$



STRATEGY ANALYSIS: STEADY STATE

Density of Ψ_t

Given that Ψ_t is normal with mean u and variance s^2, $\Psi_t \sim \mathcal{N}(u, s)$, the cumulative distribution function (CDF) F_Ψ and probability density function (PDF) f_Ψ for Ψ_t are

$$F_\Psi = \Phi\left(\frac{\psi - u}{s}\right) \tag{29}$$

$$f_\Psi = \frac{d\,F_\Psi}{d\,\psi} = \frac{1}{s}\phi\left(\frac{\psi - u}{s}\right) \tag{30}$$

where Φ is the standard normal CDF and ϕ is the standard normal PDF. Substitution of u and s from equation 28 into equation 30 gives the full density function for Ψ_t. Once the steady state regime is reached the density function has no dependence on t.

$$f_\Psi = \frac{1}{\sigma\left(\tau_2 - \tau_1\right)\sqrt{\frac{\pi}{(\tau_1+\tau_2)}}} \exp\left[\frac{-\left(\psi - \left(\mu - \frac{\sigma^2}{2}\right)\left(\tau_2 - \tau_1\right)\right)^2}{\frac{\sigma^2(\tau_2-\tau_1)^2}{(\tau_1+\tau_2)}}\right] \tag{31}$$

STRATEGY ANALYSIS: STEADY STATE

Expected Value of ϱ_t

The position indicator ϱ_t from equation 5 is a function of the random variable Ψ_t whose distribution we have from equation 28; thus, we can calculate the expected value of the position indicator using Theorem 3.2.2 part (*ix*) [Itó(1984)].

$$\begin{aligned}
\mathbb{E}\left[\varrho_t\right] &= \int_{-\infty}^{\infty} \operatorname{sgn}\left(\psi\right) f_{\Psi}(\psi)\, d\psi \\
&= \int_{-\infty}^{\infty} \operatorname{sgn}\left(\psi\right) \frac{1}{s}\phi\left(\frac{\psi-u}{s}\right)\, d\psi \\
&= \int_{-\infty}^{0} (-1)\frac{1}{s}\phi\left(\frac{\psi-u}{s}\right)\, d\psi + \int_{0}^{\infty} (1)\frac{1}{s}\phi\left(\frac{\psi-u}{s}\right)\, d\psi \\
&= \lim_{a\to\infty}\left[\Phi\left(\frac{\psi-u}{s}\right)\right]_0^a - \lim_{b\to-\infty}\left[\Phi\left(\frac{\psi-u}{s}\right)\right]_b^0 \\
&= \left[1-\Phi\left(\frac{0-u}{s}\right)\right] - \left[\Phi\left(\frac{0-u}{s}\right) - 0\right]
\end{aligned}$$

STRATEGY ANALYSIS: STEADY STATE

Expected Value of ϱ_t Continued

$$\begin{aligned}\mathbb{E}\left[\varrho_t\right] &= 1 - 2\Phi\left(-\frac{u}{s}\right)\\ &= 1 - 2\Phi\left(-\frac{\left(\mu - \frac{\sigma^2}{2}\right)\left(\tau_2 - \tau_1\right)}{\frac{\sigma^2}{2}\left(\frac{(\tau_2-\tau_1)^2}{\tau_1+\tau_2}\right)}\right)\\ &= 1 - 2\Phi\left[-\frac{2}{\sigma^2}\left(\mu - \frac{\sigma^2}{2}\right)\frac{\tau_1+\tau_2}{\tau_2-\tau_1}\right] \qquad (32)\end{aligned}$$

STRATEGY ANALYSIS: STEADY STATE

Log Return Expected Value

We have now assembled the results needed to calculate the Luxor steady state expected log return r_t as given in definition 16.

$$\begin{aligned}
\mathbb{E}\left[r_t\right] &= \mathbb{E}\left[\int_0^t \varrho_u d\left(\ln Su\right)\right] \\
&= \mathbb{E}\left[\int_0^t \varrho_u \left[\left(\mu - \frac{\sigma^2}{2}\right) du + \sigma dW_u\right]\right] \\
&= \mathbb{E}\left[\left(\mu - \frac{\sigma^2}{2}\right) \int_0^t \varrho_u \, du + \sigma \int_0^t \varrho_u \, dW_u\right] \\
&= \left(\mu - \frac{\sigma^2}{2}\right) \mathbb{E}\left[\int_0^t \varrho_u \, du\right] + \sigma \, \mathbb{E}\left[\int_0^t \varrho_u \, dW_u\right] \qquad (33)
\end{aligned}$$

STRATEGY ANALYSIS: LOG RETURN

Log Return

The position indicator ϱ_t by definition is a random sequence of the elements in the set $\{-1, 0, 1\}$ which implies

$$\int_0^t \varrho_u^2 \, du \leq t < \infty \tag{34}$$

thus ϱ_t^2 is square integrable on $[0, t]$. In addition, ϱ_t is $\mathcal{F}_t$ adapted; therefore, the Itô integral inside the second expected value in equation 33 mets the conditions of Theorem 4.3.1 [Shreve(2004)] and the integral is a martingale with an initial and expected value of zero.

Define the function g such that

$$\xi = g(\rho) = \int_0^t \rho \, du \tag{35}$$

and define the random variable Ξ_t.

$$\Xi_t = g(\varrho_t) = \int_0^t \varrho_u \, du \tag{36}$$

STRATEGY ANALYSIS: LOG RETURN

Log Return

Since the last term in equation 33 is zero, the expected value of the log return is given by the first term.

$$
\begin{aligned}
\mathbb{E}\left[r_t\right] &= \left(\mu - \frac{\sigma^2}{2}\right) \mathbb{E}\left[\int_0^t \varrho_u \, du\right] \\
&= \left(\mu - \frac{\sigma^2}{2}\right) \mathbb{E}\left[\Xi_t\right] \\
&= \left(\mu - \frac{\sigma^2}{2}\right) \int_{-\infty}^{\infty} \xi \, f_{\Xi_t}(\xi) \, d\xi \\
&\quad \text{Substitute } \xi = g(\rho) \text{ and } f_{\Xi_t}(\xi) \, d\xi = f_{\varrho_t}(\rho) \, d\rho \\
&\quad \text{via Theorem 3.2.2 part } (ix) \text{ in [Itó(1984)]} \\
&= \left(\mu - \frac{\sigma^2}{2}\right) \int_{-\infty}^{\infty} g(\rho) \, f_{\varrho_t}(\rho) \, d\rho \\
&= \left(\mu - \frac{\sigma^2}{2}\right) \int_{-\infty}^{\infty} \left(\int_0^t \rho \, du\right) f_{\varrho_t}(\rho) \, d\rho && (37) \\
& && (38)
\end{aligned}
$$

STRATEGY ANALYSIS: LOG RETURN

Log Return

Change the order of integration via Fubini's theorem

$$\begin{aligned} &= \left(\mu - \frac{\sigma^2}{2}\right) \int_0^t \left(\int_{-\infty}^{\infty} \rho f_{\varrho_t}(\rho)\, d\rho\right) du \\ &= \left(\mu - \frac{\sigma^2}{2}\right) \int_0^t \mathbb{E}\left[\varrho_u\right] du \end{aligned} \tag{39}$$

Inserting the expected value of ϱ from equation 32 into equation 39 and solving the integral gives the expected log return.

$$\begin{aligned} \mathbb{E}\left[r_t\right] &= \left(\mu - \frac{\sigma^2}{2}\right) \int_0^t \left[1 - 2\Phi\left(-\frac{2}{\sigma^2}\left(\mu - \frac{\sigma^2}{2}\right)\frac{\tau_1 + \tau_2}{\tau_2 - \tau_1}\right)\right] du \\ &= \left(\mu - \frac{\sigma^2}{2}\right)\left(1 - 2\Phi\left[-\frac{2}{\sigma^2}\left(\mu - \frac{\sigma^2}{2}\right)\frac{\tau_1 + \tau_2}{\tau_2 - \tau_1}\right]\right) t \end{aligned} \tag{40}$$

STRATEGY ANALYSIS: LOG RETURN

Proposition:

Given the assumptions and model in sections 5 and 6, the expected steady state log return of the Luxor strategy is greater than or equal to zero.

$$\mathbb{E}\left[r_t\right] \geq 0 \tag{41}$$

Proof:

We proceed by determining the sign of each term in equation 40. t and σ^2 are positive by definition, and $\frac{\tau_1+\tau_2}{\tau_2-\tau_1} > 0$ since $0 < \tau_1 < \tau_2$. Let

$$C = -\frac{2}{\sigma^2}\left(\mu - \frac{\sigma^2}{2}\right)\frac{\tau_1 + \tau_2}{\tau_2 - \tau_1} \tag{42}$$

Thus the order relation between μ and $\frac{\sigma^2}{2}$ controls the sign of C.

STRATEGY ANALYSIS: LOG RETURN

Proof (Continued):

Table 1 gives the value of μ relative to $\frac{\sigma^2}{2}$ in the left column, the sign of each term that can take on a negative value in the center columns and the resulting expected log return $\mathbb{E}\left[r_t\right]$ in the right hand column.

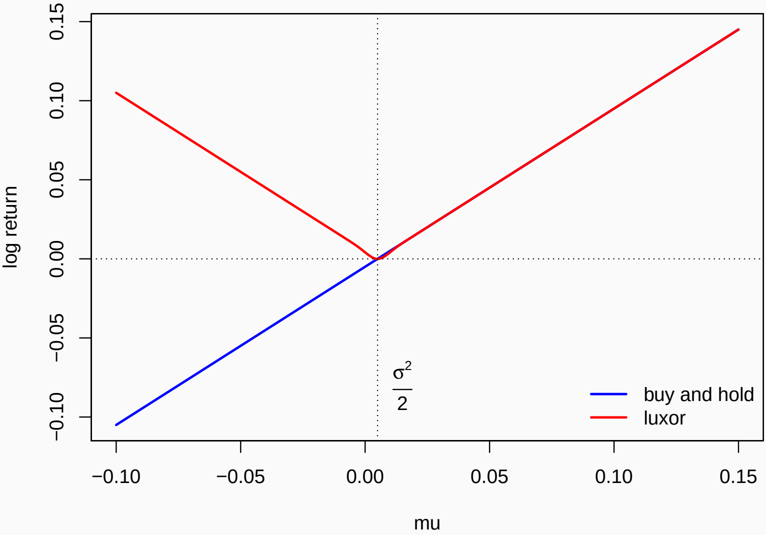
μ	C	Φ	$1 - 2\Phi$	$\left(\mu - \frac{\sigma^2}{2}\right)$	$\mathbb{E}\left[r_t\right]$
$> \frac{\sigma^2}{2}$	-	$< \frac{1}{2}$	+	+	+
$= \frac{\sigma^2}{2}$	0	$= \frac{1}{2}$	0	0	0
$< \frac{\sigma^2}{2}$	+	$> \frac{1}{2}$	-	-	+

Table 1: Expected Log Return Relative to μ and σ

Comparing the far left hand column with the far right hand column in table 1, we see that for every $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}_{\geq 0}$, and $0 < \tau_1 < \tau_2$, the expected log return r_t is greater than or equal to zero.

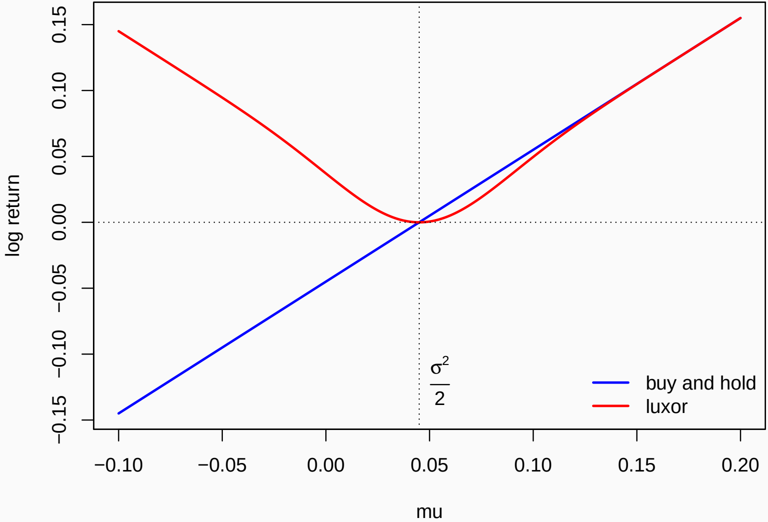
STRATEGY ANALYSIS: LOG RETURN

$$\frac{d\,\mathbb{E}[r_t]}{d\,t} = \left(\mu - \frac{\sigma^2}{2}\right)\left(1 - 2\Phi\left[-\frac{2}{\sigma^2}\left(\mu - \frac{\sigma^2}{2}\right)\frac{\tau_1+\tau_2}{\tau_2-\tau_1}\right]\right) \qquad \sigma = 0.1, \tau_1 = 3, \tau_2 = 20, t = 1$$



STRATEGY ANALYSIS: LOG RETURN

$$\frac{d\,\mathbb{E}[r_t]}{d\,t} = \left(\mu - \frac{\sigma^2}{2}\right)\left(1 - 2\Phi\left[-\frac{2}{\sigma^2}\left(\mu - \frac{\sigma^2}{2}\right)\frac{\tau_1+\tau_2}{\tau_2-\tau_1}\right]\right) \qquad \sigma = 0.3, \tau_1 = 3, \tau_2 = 20, t = 1$$



SUMMARY

SUMMARY

Results

Given the assumptions and model in sections 5 and 6, we

- Derived a closed form solution for the expected log returns of Luxor
- Demonstrated Luxor has a non negative expected value
- When $\mu > \frac{\sigma^2}{2}$ high volatility causes under performance compared to buy and hold

Issues

Issues we are currently addressing

- Simple SDE model does not capture features of stock prices that may be advantageous to strategies like Luxor
- More accurate price SDE models
- More realistic transaction and slippage models

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