Heston Stochastic Local Volatility Model

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Motivation

Combine two of the most popular option pricing models, the Local Volatility model with $x = \ln S_t$

$$dx_t = \left(r_t - q_t - \frac{\sigma_{LV}^2(x_t, t)}{2} \right) dt + \sigma_{LV}(x_t, t) dW_t$$

and the Heston Stochastic Volatility model

$$\begin{aligned} dx_t &= \left(r_t - q_t - \frac{\nu_t}{2} \right) dt + \sqrt{\nu_t} dW_t^x \\ d\nu_t &= \kappa \left(\theta - \nu_t \right) dt + \sigma \sqrt{\nu_t} dW_t^\nu \\ \rho dt &= dW_t^\nu dW_t^x \end{aligned}$$

to control the forward volatility dynamics and the calibration error.

Model Definition

Add leverage function $L(S_t, t)$ and mixing factor η to the Heston model:

$$
\begin{aligned}
dx_t &= \left(r_t - q_t - \frac{L^2(x_t, t)}{2}\nu_t \right) dt + L(x_t, t)\sqrt{\nu_t} dW_t^x \\
d\nu_t &= \kappa\left(\theta - \nu_t\right) dt + \eta\sigma\sqrt{\nu_t} dW_t^\nu \\
\rho dt &= dW_t^\nu dW_t^x
\end{aligned}
$$

Leverage $L(x_t, t)$ is given by probability density $p(x_t, \nu, t)$ and

$$
L(x_t, t) = \frac{\sigma_{LV}(x_t, t)}{\sqrt{\mathbb{E}[\nu_t | x = x_t]}} = \sigma_{LV}(x_t, t)\sqrt{\frac{\int_{\mathbb{R}^+} p(x_t, \nu, t) d\nu}{\int_{\mathbb{R}^+} \nu p(x_t, \nu, t) d\nu}}
$$

Mixing factor η tunes between stochastic and local volatility

Package RHestonSLV: Calibration and Pricing

Calibration:

- Calculate Heston parameters $\{\kappa, \theta, \sigma, \rho, v_{t=0}\}$ and $\sigma_{LV}(x_t, t)$
- Compute $p(x_t, \nu, t)$ either by Monte-Carlo or PDE to get to the leverage function $L_t(x_t, t)$
- Infer the mixing factor η from prices of exotic options

Package HestonSLV

✓ Monte-Carlo and PDE calibration

✓ Pricing of vanillas and exotic options like double-no-touch barriers

✓ Implementation is based on QuantLib, www.quantlib.org

Cheat Sheet: Link between SDE and PDE

Starting point is a linear, multidimensional SDE of the form:

$$d\boldsymbol{x}_t = \boldsymbol{\mu}(\boldsymbol{x}_t, t)dt + \boldsymbol{\sigma}(\boldsymbol{x}_t, t)d\boldsymbol{W}_t$$

Feynman-Kac: the price of a derivative $u(\boldsymbol{x}_t, t)$ with boundary condition $u(\boldsymbol{x}_T, T)$ at maturity T is given by:

$$\partial_t u + \sum_{k=1}^{n} \mu_i \partial_{x_k} u + \frac{1}{2} \sum_{k,l=1}^{n} \left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)_{kl} \partial_{x_k}\partial_{x_l} u - ru = 0$$

Fokker-Planck: the time evolution of the probability density function $p(\boldsymbol{x}_t, t)$ with the initial condition $p(\boldsymbol{x}, t = 0) = \delta(\boldsymbol{x} - \boldsymbol{x_0})$ is given by:

$$\partial_t p = -\sum_{k=1}^{n} \partial_{x_k}\left[\mu_i p\right] + \frac{1}{2} \sum_{k,l=1}^{n} \partial_{x_k}\partial_{x_l}\left[\left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)_{kl} p\right]$$

Fokker-Planck Forward Equation

The corresponding Fokker-Planck equation for the probability density $p : \mathbb{R} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, (x, \nu, t) \mapsto p(x, \nu, t)$ is:

$$\begin{aligned}\partial_t p &= \frac{1}{2}\partial_x^2 \left[L^2 \nu p\right] + \frac{1}{2}\eta^2\sigma^2\partial_\nu^2 \left[\nu p\right] + \eta\sigma\rho\partial_x\partial_\nu \left[L\nu p\right] \\ &\quad -\partial_x \left[\left(r - q - \frac{1}{2}L^2\nu\right) p\right] - \partial_\nu \left[\kappa\left(\theta - \nu\right) p\right]\end{aligned}$$

Numerical solution of the PDE is cumbersome due to difficult boundary conditions and the δ-distribution as the initial condition.

PDE can be efficiently solved by using operator splitting schemes, preferable the modified Craig-Sneyd scheme.

Calibration: Fokker-Planck Forward Equation

- ✓ Zero-Flux boundary condition for $\nu = \{\nu_{min}, \nu_{max}\}$
- ✓ Reformulate PDE in terms of $q = \nu^{1-\frac{2\kappa\theta}{\sigma^2}}$ or $z = \ln \nu$ if the Feller constraint is violated
- ✓ Prediction-Correction step for $L(x_{t+\Delta t}, t + \Delta t)$
- ✓ Non-uniform grids are a key factor for success
- ✓ Includes adaptive time step size and grid boundaries to allow for rapid changes of the shape of $p(x_t, \nu, t)$ for small t
- ✓ Semi-analytical approximations of initial δ-distribution for small t

Corresponding Feynman-Kac backward PDE is much easier to solve.

Calibration: Monte-Carlo Simulation

The quadratic exponential discretization can be adapted to simulate the Heston SLV model efficiently.

Reminder: $L(x_t, t) = \frac{\sigma_{LV}(x_t,t)}{\sqrt{\mathbb{E}[\nu_t|x=x_t]}}$

1. Simulate the next time step for all calibration paths
2. Define set of n bins $b_i = \{x_t^i, x_t^i + \Delta x_t^i\}$ and assign paths to bins
3. Calculate expectation value $e_i = \mathbb{E}[\nu_t|x \in b_i]$ over all paths in b_i
4. Define $L(x_t \in b_i, t) = \frac{\sigma_{LV}(x_t,t)}{e_i}$
5. $t \leftarrow t + \Delta t$ and goto 1

Advice: Use Quasi-Monte-Carlo simulations with Brownian bridges.

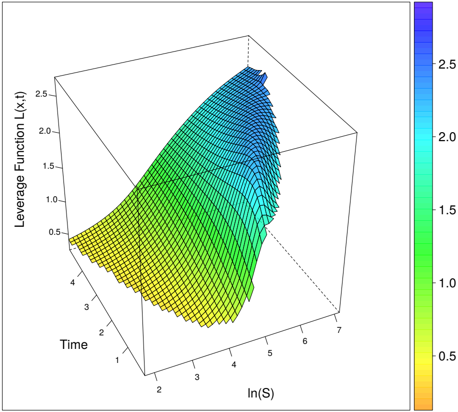
Calibration: Test Bed

Motivation: Set-up extreme test case for the SLV calibration

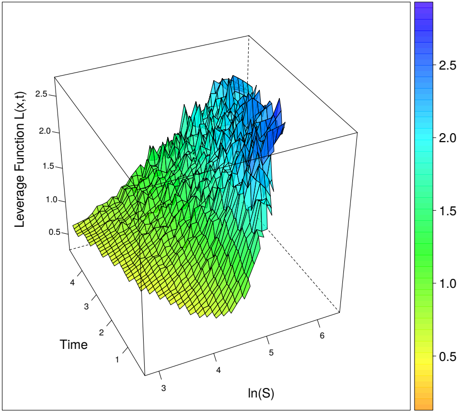
- Local Volatility: $\sigma_{LV}(x,t) \equiv 30\%$
- Heston parameters:
  $S_0 = 100, \nu_0 = 0.09, \kappa = 1.0, \theta = 0.06, \sigma = 0.4, \rho = -75\%$
- Feller condition is violated with $\frac{2\kappa\theta}{\sigma^2} = 0.75$
- Implied volatility surface of the Heston model and the Local Volatility model differ significantly.

Calibration: Fokker-Planck PDE vs Monte-Carlo

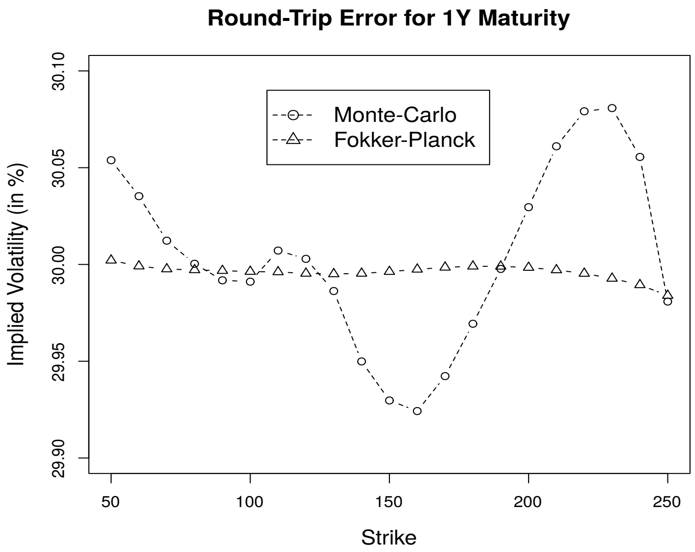
Fokker-Planck Forward Equation, η=1.00



Monte-Carlo Simulation, η=1.00

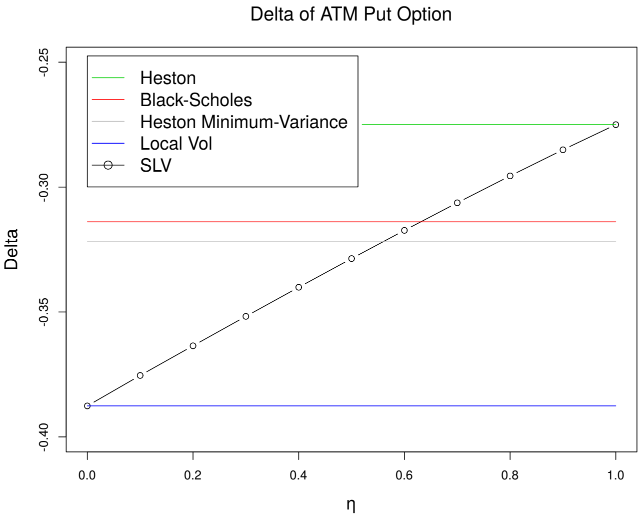


Calibration Sanity Check: Round-Trip Error for Vanillas



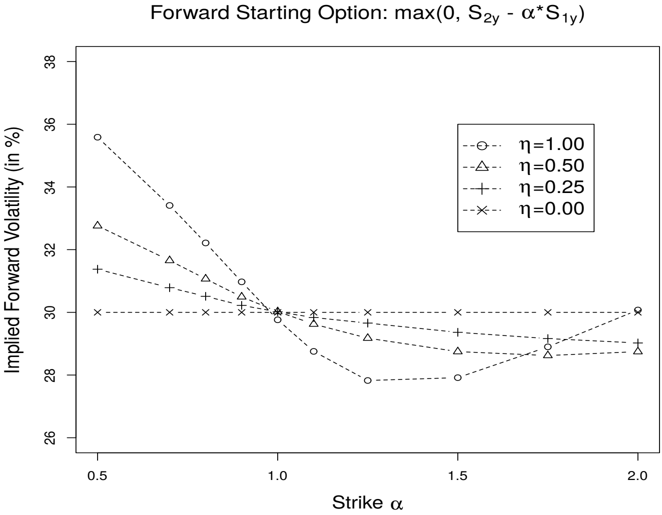
Case Study: Delta of Vanilla Option

Vanilla Put Option: 3y maturity, S_0=100, strike=100



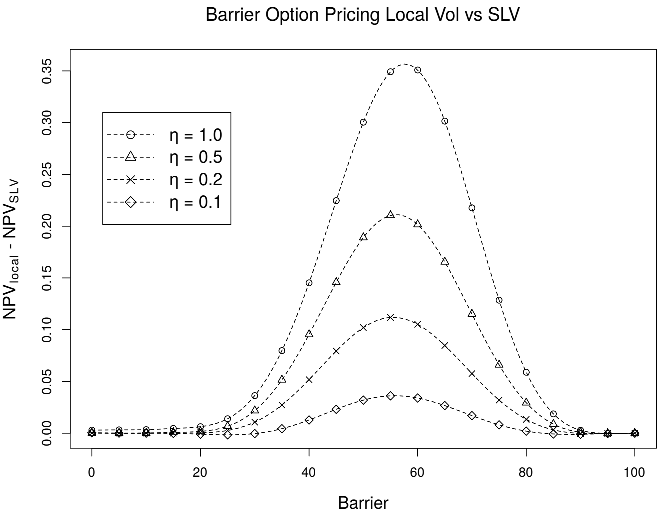
Choose the Forward Volatility Skew Dynamics

Interpolate between the Local and the Heston skew dynamics by tuning η between 0 and 1.



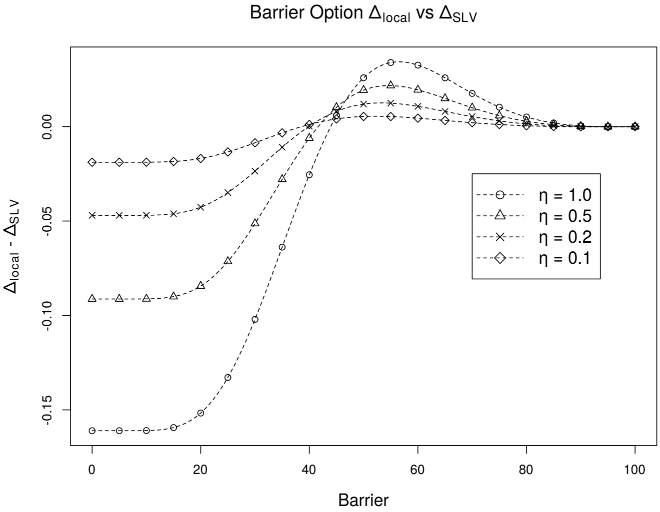
Case Study: Barrier Option Prices

DOP Barrier Option: 3y maturity, S_0=100, strike=100



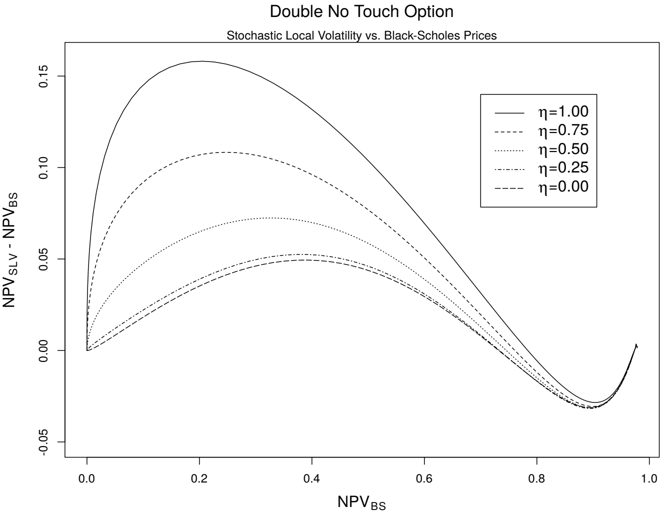
Case Study: Delta of Barrier Options

DOP Barrier Option: 3y maturity, S_0=100, strike=100



Case Study: Double-No-Touch Options

Knock-Out Double-No-Touch Option: 1y maturity, S_0=100



Summary: Heston Stochastic Local Volatility

- RHestonSLV: A package for the Heston Stochastic Local Volatility Model
- Monte-Carlo Calibration
- Calibration via Fokker-Planck Forward Equation
- Supports pricing of vanillas and exotic options
- Implementation is based on QuantLib 1.8 and Rcpp
- Package source code including all examples shown is on github
  `https://github.com/klausspanderen/RHestonSLV`

Literature

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