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Motivation

Combine two of the most popular option pricing models, the Local Volatility model with $x = \ln S_t$

$$dx_t = \left( r_t - q_t - \frac{\sigma^2_{LV}(x_t, t)}{2} \right) dt + \sigma_{LV}(x_t, t)dW_t$$

and the Heston Stochastic Volatility model

$$dx_t = \left( r_t - q_t - \frac{\nu_t}{2} \right) dt + \sqrt{\nu_t}dW^x_t$$
$$d\nu_t = \kappa (\theta - \nu_t) dt + \sigma \sqrt{\nu_t}dW^\nu_t$$
$$\rho dt = dW^\nu_t dW^x_t$$

to control the forward volatility dynamics and the calibration error.
Add leverage function $L(S_t, t)$ and mixing factor $\eta$ to the Heston model:

$$
dx_t = \left( r_t - q_t - \frac{L^2(x_t, t)}{2} \nu_t \right) dt + L(x_t, t) \sqrt{\nu_t} dW_t^x
$$

$$
d\nu_t = \kappa (\theta - \nu_t) dt + \eta \sigma \sqrt{\nu_t} dW_t^\nu
$$

$$
\rho dt = dW_t^\nu dW_t^x
$$

Leverage $L(x_t, t)$ is given by probability density $p(x_t, \nu, t)$ and

$$
L(x_t, t) = \frac{\sigma_{LV}(x_t, t)}{\sqrt{\mathbb{E}[\nu_t | x = x_t]}} = \sigma_{LV}(x_t, t) \sqrt{\frac{\int_{\mathbb{R}^+} p(x_t, \nu, t) d\nu}{\int_{\mathbb{R}^+} \nu p(x_t, \nu, t) d\nu}}
$$

Mixing factor $\eta$ tunes between stochastic and local volatility
Calibration:

- Calculate Heston parameters $\{\kappa, \theta, \sigma, \rho, \nu_{t=0}\}$ and $\sigma_{LV}(x_t, t)$
- Compute $p(x_t, \nu, t)$ either by Monte-Carlo or PDE to get to the leverage function $L_t(x_t, t)$
- Infer the mixing factor $\eta$ from prices of exotic options

**Package HestonSLV**

- Monte-Carlo and PDE calibration
- Pricing of vanillas and exotic options like double-no-touch barriers
- Implementation is based on QuantLib, www.quantlib.org
Cheat Sheet: Link between SDE and PDE

Starting point is a linear, multidimensional SDE of the form:

$$d\mathbf{x}_t = \mu(\mathbf{x}_t, t)dt + \sigma(\mathbf{x}_t, t)dW_t$$

**Feynman-Kac:** the price of a derivative $u(\mathbf{x}_t, t)$ with boundary condition $u(\mathbf{x}_T, T)$ at maturity $T$ is given by:

$$\partial_t u + \sum_{k=1}^n \mu_i \partial_{x_k} u + \frac{1}{2} \sum_{k,l=1}^n \left( \sigma \sigma^T \right)_{kl} \partial_{x_k} \partial_{x_l} u - ru = 0$$

**Fokker-Planck:** the time evolution of the probability density function $p(\mathbf{x}_t, t)$ with the initial condition $p(\mathbf{x}, t = 0) = \delta(\mathbf{x} - \mathbf{x}_0)$ is given by:

$$\partial_t p = -\sum_{k=1}^n \partial_{x_k} [\mu_i p] + \frac{1}{2} \sum_{k,l=1}^n \partial_{x_k} \partial_{x_l} \left[ \left( \sigma \sigma^T \right)_{kl} p \right]$$
The corresponding Fokker-Planck equation for the probability density
\( p : \mathbb{R} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}, (x, \nu, t) \mapsto p(x, \nu, t) \) is:

\[
\frac{\partial_t p}{2} = \frac{1}{2} \frac{\partial^2}{\partial x^2} \left[ L^2 \nu p \right] + \frac{1}{2} \eta^2 \sigma^2 \frac{\partial^2}{\partial \nu^2} \left[ \nu p \right] + \eta \sigma \rho \frac{\partial}{\partial x} \frac{\partial}{\partial \nu} \left[ L \nu p \right] \\
- \frac{\partial}{\partial x} \left[ \left( r - q - \frac{1}{2} L^2 \nu \right) p \right] - \frac{\partial}{\partial \nu} \left[ \kappa (\theta - \nu) p \right]
\]

Numerical solution of the PDE is cumbersome due to difficult boundary conditions and the \( \delta \)-distribution as the initial condition.

PDE can be efficiently solved by using operator splitting schemes, preferable the modified Craig-Sneyd scheme.
Zero-Flux boundary condition for \( \nu = \{\nu_{\text{min}}, \nu_{\text{max}}\} \)

Reformulate PDE in terms of \( q = \nu^{1-\frac{2\kappa\theta}{\sigma^2}} \) or \( z = \ln \nu \) if the Feller constraint is violated

Prediction-Correction step for \( L(x_{t+\Delta t}, t + \Delta t) \)

Non-uniform grids are a key factor for success

Includes adaptive time step size and grid boundaries to allow for rapid changes of the shape of \( p(x_t, \nu, t) \) for small \( t \)

Semi-analytical approximations of initial \( \delta \)-distribution for small \( t \)

Corresponding Feynman-Kac backward PDE is much easier to solve.
The quadratic exponential discretization can be adapted to simulate the Heston SLV model efficiently.

Reminder: \[ L(x_t, t) = \frac{\sigma_{LV}(x_t, t)}{\sqrt{\mathbb{E}[\nu_t | x=x_t]}} \]

1. Simulate the next time step for all calibration paths
2. Define set of \( n \) bins \( b_i = \{ x_t^i, x_t^i + \Delta x_t^i \} \) and assign paths to bins
3. Calculate expectation value \( e_i = \mathbb{E}[\nu_t | x \in b_i] \) over all paths in \( b_i \)
4. Define \( L(x_t \in b_i, t) = \frac{\sigma_{LV}(x_t, t)}{e_i} \)
5. \( t \leftarrow t + \Delta t \) and goto 1

Advice: Use Quasi-Monte-Carlo simulations with Brownian bridges.
Motivation: Set-up extreme test case for the SLV calibration

- **Local Volatility**: $\sigma_{LV}(x, t) \equiv 30\%$
- **Heston parameters**:
  
  $S_0 = 100, \nu_0 = 0.09, \kappa = 1.0, \theta = 0.06, \sigma = 0.4, \rho = -75\%$
- **Feller condition is violated** with $\frac{2\kappa\theta}{\sigma^2} = 0.75$
- **Implied volatility surface** of the Heston model and the Local Volatility model differ significantly.
Calibration: Fokker-Planck PDE vs Monte-Carlo

Fokker-Planck Forward Equation, $\eta=1.00$

Monte-Carlo Simulation, $\eta=1.00$
Calibration Sanity Check: Round-Trip Error for Vanillas

Round-Trip Error for 1Y Maturity

- - ○ - - Monte-Carlo
- - △ - - Fokker-Planck

Strike

Implied Volatility (in %)

50 100 150 200 250
29.90 29.95 30.00 30.05 30.10

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Heston Stochastic Local Volatility Model
2016-05-20
Vanilla Put Option: 3y maturity, $S_0=100$, strike=100

Delta of ATM Put Option

- Heston
- Black-Scholes
- Heston Minimum-Variance
- Local Vol
- SLV
Choose the Forward Volatility Skew Dynamics

Interpolate between the Local and the Heston skew dynamics by tuning $\eta$ between 0 and 1.

**Forward Starting Option:** $\max(0, S_{2y} - \alpha S_{1y})$

![Graph showing implied forward volatility for different values of $\eta$.](image-url)
Case Study: Barrier Option Prices

DOP Barrier Option: 3y maturity, $S_0=100$, strike=100

Barrier Option Pricing Local Vol vs SLV

Barrier

NPV$_{local} -$ NPV$_{SLV}$

$\eta = 1.0$
$\eta = 0.5$
$\eta = 0.2$
$\eta = 0.1$

Barrier

Barrier Option Pricing Local Vol vs SLV

Barrier

NPV$_{local} -$ NPV$_{SLV}$

$\eta = 1.0$
$\eta = 0.5$
$\eta = 0.2$
$\eta = 0.1$
Case Study: Delta of Barrier Options

DOP Barrier Option: 3y maturity, $S_0=100$, strike=100

Barrier Option $\Delta_{\text{local}}$ vs $\Delta_{\text{SLV}}$

Barrier $\Delta_{\text{local}} - \Delta_{\text{SLV}}$ vs $\eta$

- $\eta = 1.0$
- $\eta = 0.5$
- $\eta = 0.2$
- $\eta = 0.1$
Case Study: Double-No-Touch Options

Knock-Out Double-No-Touch Option: 1y maturity, $S_0=100$

Double No Touch Option

Stochastic Local Volatility vs. Black-Scholes Prices

η = 1.00
η = 0.75
η = 0.50
η = 0.25
η = 0.00
RHestonSLV: A package for the Heston Stochastic Local Volatility Model

Monte-Carlo Calibration

Calibration via Fokker-Planck Forward Equation

Supports pricing of vanillas and exotic options

Implementation is based on QuantLib 1.8 and Rcpp

Package source code including all examples shown is on github https://github.com/klausspanderen/RHestonSLV

