R package **mcrp**: Multiple criteria risk contribution optimization

Bernhard Pfaff  
bernhard_pfaff@fra.invesco.com

Invesco Asset Management GmbH  
Frankfurt am Main

R in Finance, Chicago IL, 19 and 20 May 2017
The concept of risk parity (aka ERC) is due to Qian (2005, 2006, 2011) (see also Maillard et al., 2010; Roncalli, 2013). Hereby, the contributions with respect to the portfolio standard deviation risk of the constituents are equal. Recently, this concept has been extended to include the higher-moment risk contributions (see Baitinger et al., 2017), i.e., skewness and kurtosis.

In this talk:
1. Optimization problem of multiple criteria risk contributions.
2. Structure of the R package mcrp.
3. Empirical application to a multi-asset class portfolio.
Multiple criteria risk optimization

Higher Moments: Definitions

Definitions provided in Jondeau and Rockinger (2006):

\[ \sigma_p^2 = \mathbb{E} \left[ \sum_{i=1}^{n} \omega_i (R_i - \mu_i)(r_p - \mu_p) \right] = \omega' \Sigma_p \]

\[ = \omega' M_2 \omega \]

\[ s_p^3 = \mathbb{E} \left[ \sum_{i=1}^{n} \omega_i (R_i - \mu_i)(r_p - \mu_p)^2 \right] = \omega' S_p \]

\[ = \omega' M_3 (\omega \otimes \omega) \]

\[ \kappa_p^4 = \mathbb{E} \left[ \sum_{i=1}^{n} \omega_i (R_i - \mu_i)(r_p - \mu_p)^3 \right] = \omega' K_p \]

\[ = \omega' M_4 (\omega \otimes \omega \otimes \omega) \]
Multiple criteria risk optimization

Higher Moments: Partial Derivatives

\[ \text{MRC}_2 = \frac{\delta \sigma_p^2}{\delta \omega} = 2M_2 \omega \]

\[ \text{MRC}_3 = \frac{S_p^3}{\delta \omega} = 3M_3(\omega \otimes \omega) \]

\[ \text{MRC}_4 = \frac{\kappa_p^4}{\delta \omega} = 4M_4(\omega \otimes \omega \otimes \omega) \]

Matrices \((n \times n)\) \(M_2\), \((n \times n^2)\) \(M_3\), \((n \times n^3)\) \(M_4\) are the centred (tensor) product moment matrices:

\[ M_2 = E\left[ (R - \mu)(R - \mu)' \right] = \{ \sigma_{ij} \} \]

\[ M_3 = E\left[ (R - \mu)(R - \mu)' \otimes (R - \mu)' \right] = \{ S_{ijk} \} \]

\[ M_4 = E\left[ (R - \mu)(R - \mu)' \otimes (R - \mu)' \otimes (R - \mu)' \right] = \{ \kappa_{ijkl} \} \]
Multiple criteria risk optimization

Problem formulation

minimize $F(\omega) = \lambda_1 \text{VAR}(ARC_2) + \lambda_2 \text{VAR}(ARC_3) + \lambda_3 \text{VAR}(ARC_4)$

subject to $\sum_{i=1}^{n} \omega_i = 1$

$0 \leq \omega_i \leq 1$ for $i = 1, \ldots, n$. 
R package **mcrp**

**Structure**

- Package is purely written in R.
- Dependencies to **FRAPO** (see Pfaff, 2016) and suggests **testthat** (see Wickham, 2011) for unit testing.
- Optimization conducted with `stats::nlmib()`.
- Core function: `mcrp()`.
- Auxiliary functions: `Portfoo()`, `PortfooDeriv()`, `PortfooContrib()` with `foo = {Risk, Skew, Kurt}`.
- Available on GitHub: [https://github.com/bpfaff/mcrp](https://github.com/bpfaff/mcrp)
R package **mcrp**

Core function: Input

**Function mcrp()**

```r
> args(mcrp)

function (start, returns, lambda = c(1, 1, 1), ...)
NULL
```

**Arguments**

- **start**: vector of starting values.
- **returns**: matrix of assets’ returns.
- **lambda**: selection/weighting of sub-objectives; set element(s) to NA for exclusion.
- **...**: ellipsis argument is passed down to stats::nlmnb().
R package **mcrp**

Core function: Output

---

**S4-Class: PortSol from package FRAPO**

```r
> showClass("PortSol")

Class "PortSol" [package "FRAPO"]

Slots:

Name:  weights  opt  type  call
Class: numeric  list  character  call

Known Subclasses: "PortCdd", "PortAdd", "PortMdd"
```

**Slots**

- **weights**: portfolio weight vector.
- **opt**: list object returned by `stats::nlmnib()`.
- **type**: description of portfolio problem.
- **call**: the call to `mcrp()` (used for `stats::update()`-method).
Empirical application

Specification

- Fix-me strategy:
  1. 1st sub-sample until 12/2008 used for optimization.
  2. 2nd sub-sample from 01/2009 for backtest.
- Portfolio optimizations with respect to:
  1. single criteria: contrib. to risk.
  2. single criteria: contrib. to skewness.
  3. single criteria: contrib. to kurtosis.
  4. multiple criteria: contrib. to risk and skewness.
  5. multiple criteria: contrib. to risk and kurtosis.
  6. multiple criteria: contrib. to risk, skewness and kurtosis.
Empirical application

R code: Data and descriptive statistics

```r
> library(mcrp)
> data(MultiAsset)
> P <- as.timeSeries(MultiAsset[, c("GSPC", "GDAXI", "FTSE", "EEM", "GLD")])
> R <- returns(P) * 100
> K <- ncol(R)
> ew <- rep(1 / K, K) ## equal weight / starting values
> ans <- cbind(apply(R, 2, sd),
+               apply(R, 2, skewness),
+               apply(R, 2, kurtosis, method = "moment"))

<table>
<thead>
<tr>
<th>Index</th>
<th>Risk</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>GSPC</td>
<td>4.835</td>
<td>-0.897</td>
<td>4.876</td>
</tr>
<tr>
<td>GDAXI</td>
<td>5.980</td>
<td>-1.002</td>
<td>5.096</td>
</tr>
<tr>
<td>FTSE</td>
<td>4.377</td>
<td>-0.704</td>
<td>3.581</td>
</tr>
<tr>
<td>EEM</td>
<td>8.079</td>
<td>-0.744</td>
<td>4.380</td>
</tr>
<tr>
<td>GLD</td>
<td>5.474</td>
<td>-0.485</td>
<td>3.852</td>
</tr>
</tbody>
</table>

Table: Empirical higher-moments
```
Empirical application

R code: Portfolio optimizations

```r
> ## single criteria: contrib. to risk
> p100 <- mcrp(ew, R, lambda = c(1, NA, NA), lower = 0)
> ## single criteria: contrib. to skewness
> p010 <- mcrp(ew, R, lambda = c(NA, 1, NA), lower = 0)
> ## single criteria: contrib. to kurtosis
> p001 <- mcrp(ew, R, lambda = c(NA, NA, 1), lower = 0)
> ## multiple criteria: contrib. to risk and skewness
> p110 <- mcrp(ew, R, lambda = c(1, 1, NA), lower = 0)
> ## multiple criteria: contrib. to risk and kurtosis
> p101 <- mcrp(ew, R, lambda = c(1, NA, 1), lower = 0)
> ## multiple criteria: contrib. to risk, skewness and kurtosis
> p111 <- mcrp(ew, R, lambda = c(1, 1, 1), lower = 0)
> ## aggregating results to list object
> popt <- list(p100, p010, p001, p110, p101, p111)
> pn <- length(popt)
> pnames <- c("p100", "p010", "p001", "p110", "p101", "p111")
> names(popt) <- pnames
> ## checking solutions
> (sopt <- unlist(lapply(popt, function(x) Solution(x)$convergence)))

p100 p010 p001 p110 p101 p111
 0   0   0   0   0   0
```
Empirical application

R code: Allocations

```r
> wopt <- lapply(popt, Weights)
> wmat <- matrix(unlist(wopt), nrow = K, ncol = pn)
> colnames(wmat) <- pnames
> rownames(wmat) <- colnames(P)
> ans <- round(wmat * 100, 2)
```

<table>
<thead>
<tr>
<th>Index</th>
<th>p100</th>
<th>p010</th>
<th>p001</th>
<th>p110</th>
<th>p101</th>
<th>p111</th>
</tr>
</thead>
<tbody>
<tr>
<td>GSPC</td>
<td>19.47</td>
<td>17.43</td>
<td>18.16</td>
<td>18.09</td>
<td>18.77</td>
<td>18.07</td>
</tr>
<tr>
<td>FTSE</td>
<td>22.13</td>
<td>31.99</td>
<td>30.13</td>
<td>30.27</td>
<td>28.76</td>
<td>30.21</td>
</tr>
<tr>
<td>EEM</td>
<td>10.99</td>
<td>12.18</td>
<td>11.60</td>
<td>12.42</td>
<td>11.87</td>
<td>12.01</td>
</tr>
<tr>
<td>GLD</td>
<td>30.17</td>
<td>16.97</td>
<td>18.66</td>
<td>17.80</td>
<td>19.58</td>
<td>18.25</td>
</tr>
</tbody>
</table>

Table: Optimal allocations
Empirical application

R code: In-sample characteristics, part I

```r
> ## Function for higher moments and portfolio contributions
> momis <- function(w, r = R){
+   a1 <- PortRisk(r, w)
+   a2 <- PortSkew(r, w)
+   a3 <- PortKurt(r, w)
+   a4 <- sd(PortRiskContrib(r, w))
+   a5 <- sd(PortSkewContrib(r, w))
+   a6 <- sd(PortKurtContrib(r, w))
+   a7 <- mean(c(a4, a5, a6))
+   a <- c(a1, a2, a3, a4, a5, a6, a7)
+   a
+ }
> ans <- matrix(unlist(lapply(wopt, momis)), ncol = pn)
> colnames(ans) <- pnames
> rownames(ans) <- c("Risk", "Skewness", "Kurtosis",
>                     "Sd Risk ctrb.", "Sd Skew ctrb.", "Sd Kurt ctrb.",
>                     "Average of Sd")
```

Pfaff (Invesco)
## Empirical application

**R code: In-sample characteristics, part II**

<table>
<thead>
<tr>
<th>Measure</th>
<th>p100</th>
<th>p010</th>
<th>p001</th>
<th>p110</th>
<th>p101</th>
<th>p111</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Moments</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Skewness</td>
<td>−1.042</td>
<td>−0.889</td>
<td>−0.908</td>
<td>−0.900</td>
<td>−0.920</td>
<td>−0.904</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>5.961</td>
<td>4.589</td>
<td>4.740</td>
<td>4.684</td>
<td>4.849</td>
<td>4.712</td>
</tr>
<tr>
<td><strong>Contributions</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sd Risk ctrb.</td>
<td>0.000</td>
<td>0.088</td>
<td>0.078</td>
<td>0.081</td>
<td>0.071</td>
<td>0.080</td>
</tr>
<tr>
<td>Sd Skew ctrb.</td>
<td>0.120</td>
<td>0.000</td>
<td>0.018</td>
<td>0.012</td>
<td>0.029</td>
<td>0.014</td>
</tr>
<tr>
<td>Sd Kurt ctrb.</td>
<td>0.100</td>
<td>0.017</td>
<td>0.000</td>
<td>0.009</td>
<td>0.012</td>
<td>0.005</td>
</tr>
<tr>
<td>Average of Sd</td>
<td>0.073</td>
<td>0.035</td>
<td>0.032</td>
<td>0.034</td>
<td>0.037</td>
<td>0.033</td>
</tr>
</tbody>
</table>

**Table:** In-sample characteristics
Summary

- Approach for an extended portfolio risk balancing with respect to higher moments.
- Weighting between (higher) moment risk contributions is possible by setting of $\lambda$, and hereby allowing selection of certain kind of risk contributions as special cases.
- Empirical example: By considering higher moment risk contributions, overall dispersion is reduced compared to the ERC-only solution.
- Caveats:
  1. Casting of optimization problem not ideal, if constituents have differing signs for skewness.
  2. No guarantee that risk contributions of higher moment risks are the same; but at least a solution of least dispersed risk contributions is obtained.
Bibliography I


