# NEAREST COMOMENT ESTIMATION WITH UNOBSERVED FACTORS AND LINEAR SHRINKAGE

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# OUTLINE

# Higher Order Moments

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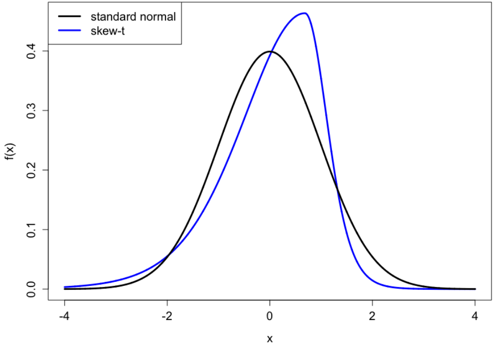
## UNIVARIATE MOMENTS

The moments describe the characteristics of the univariate distribution of portfolio returns $r_p$.

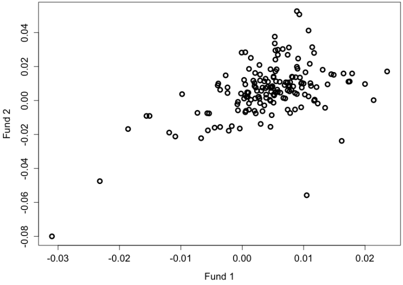
- mean, $\mu = \mathbb{E}[r_p]$
- variance, $\sigma^2 = \mathbb{E}[(r_p - \mu)^2]$
- skewness, $\phi = \mathbb{E}[(r_p - \mu)^3]$
- kurtosis, $\psi = \mathbb{E}[(r_p - \mu)^4]$



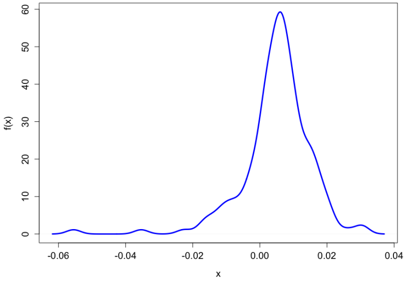
# UNIVARIATE VS. MULTIVARIATE

It holds that $r_p = \boldsymbol{w}'\boldsymbol{X}$,
$\boldsymbol{X}$ a $p$-dimensional random variable of asset returns.

(a) Sample of $\boldsymbol{X}$

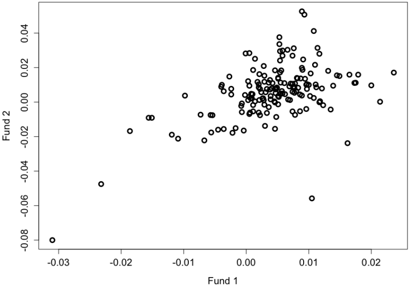



(b) $r_p = 0.5X_1 + 0.5X_2$



# MULTIVARIATE MOMENTS

## COVARIANCE MATRIX

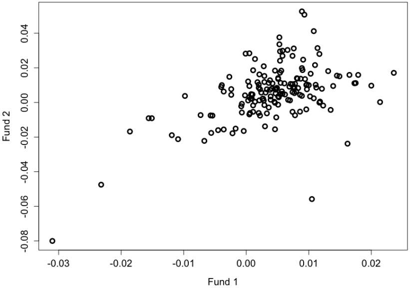



Covariance between 2 assets: $\sigma_{ij} = \mathbb{E}[(X_i - \mu_i)(X_j - \mu_j)]$

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = \begin{pmatrix} 5.344 & 5.937 \\ 5.937 & 25.181 \end{pmatrix} \times 10^{-5}, \qquad (\rho = 0.512)$$

# MULTIVARIATE MOMENTS

## COSKEWNESS MATRIX




Coskewness between 3 assets:

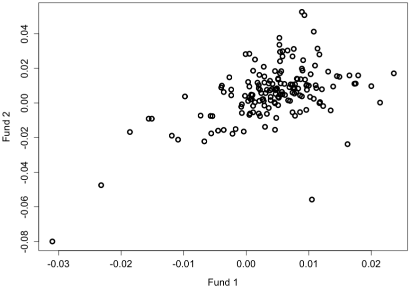
$$\phi_{ijk} = \mathbb{E}[(X_i - \mu_i)(X_j - \mu_j)(X_k - \mu_k)]$$

$$\mathbf{\Phi} = \begin{pmatrix} \phi_{111} & \boldsymbol{\phi_{112}} & \phi_{211} & \phi_{212} \\ \phi_{121} & \phi_{122} & \underline{\phi_{221}} & \phi_{222} \end{pmatrix}$$

$$= \begin{pmatrix} -0.458 & \mathbf{-1.107} & -1.107 & -1.950 \\ -1.107 & -1.950 & \underline{-1.950} & -4.522 \end{pmatrix} \times 10^{-6}$$

# MULTIVARIATE MOMENTS

## COKURTOSIS MATRIX




Cokurtosis between 4 assets:

$$\psi_{ijkl} = \mathbb{E}[(X_i - \mu_i)(X_j - \mu_j)(X_k - \mu_k)(X_l - \mu_l)]$$

$$\mathbf{\Psi} = \begin{pmatrix} \color{red}{\psi_{1111}} & \boldsymbol{\psi_{1112}} & \psi_{1211} & \underline{\boldsymbol{\psi_{1212}}} & \psi_{2111} & \psi_{2112} & \psi_{2211} & \psi_{2212} \\ \psi_{1121} & \psi_{1122} & \psi_{1221} & \psi_{1222} & \psi_{2121} & \psi_{2122} & \underline{\psi_{2221}} & \color{blue}{\psi_{2222}} \end{pmatrix}$$

$$= \begin{pmatrix} \color{red}{0.207} & \mathbf{0.351} & 0.351 & \underline{\mathbf{0.785}} & 0.351 & 0.785 & 0.785 & 1.678 \\ 0.351 & 0.785 & 0.785 & 1.678 & 0.785 & 1.678 & \underline{1.678} & \color{blue}{5.799} \end{pmatrix}$$

## WHY BOTHER?

1. Easy to compute portfolio moments:

$$\sigma_{r_p}^2 = \boldsymbol{w}' \boldsymbol{\Sigma}_{\boldsymbol{X}} \boldsymbol{w}$$

$$\phi_{r_p} = \boldsymbol{w}' \boldsymbol{\Phi}_{\boldsymbol{X}} (\boldsymbol{w} \otimes \boldsymbol{w})$$

$$\psi_{r_p} = \boldsymbol{w}' \boldsymbol{\Psi}_{\boldsymbol{X}} (\boldsymbol{w} \otimes \boldsymbol{w} \otimes \boldsymbol{w})$$

2. Risk budgets for portfolio optimization, e.g. mVaR, mES (Zangari (1996), Boudt, Peterson and Croux (2008))

$$\text{mVaR}(\alpha) = \mu_{r_p} + \sigma_{r_p}(z_\alpha + c_1 \boldsymbol{s}_{r_p} + c_2 \kappa_{r_p} - c_3 \boldsymbol{s}_{r_p}{}^2),$$

with $s_{r_p} = \phi_{r_p} / \sigma_{r_p}^3$ and $\kappa_{r_p} = \psi_{r_p} / \sigma_{r_p}^4 - 3$.

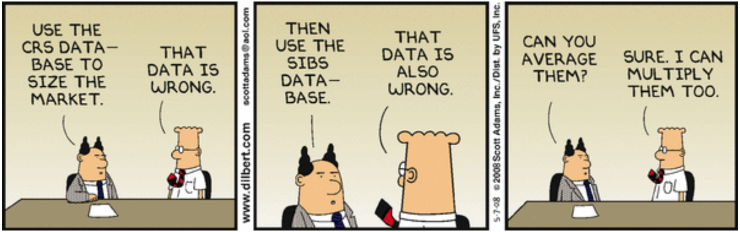
3. Discover variance, skewness and kurtosis hotspots in the portfolio, hedge when necessary. (Rubinstein (1973), Diacogiannis(1995), Litterman (1996))
4. Gradient-based optimization algorithms

# Estimation

# THEORY VS. PRACTICE

In practice: **estimation**

**Garbage in, garbage out**



# SAMPLE ESTIMATORS

Sample estimators:

$$\widehat{\Sigma}_s = \frac{1}{n}\sum_{i=1}^{n}(\boldsymbol{x}_i - \overline{\boldsymbol{x}})(\boldsymbol{x}_i - \overline{\boldsymbol{x}})',$$

$$\widehat{\Phi}_s = \frac{1}{n}\sum_{i=1}^{n}(\boldsymbol{x}_i - \overline{\boldsymbol{x}})(\boldsymbol{x}_i - \overline{\boldsymbol{x}})' \otimes (\boldsymbol{x}_i - \overline{\boldsymbol{x}})',$$

$$\widehat{\Psi}_s = \frac{1}{n}\sum_{i=1}^{n}(\boldsymbol{x}_i - \overline{\boldsymbol{x}})(\boldsymbol{x}_i - \overline{\boldsymbol{x}})' \otimes (\boldsymbol{x}_i - \overline{\boldsymbol{x}})' \otimes (\boldsymbol{x}_i - \overline{\boldsymbol{x}})'.$$

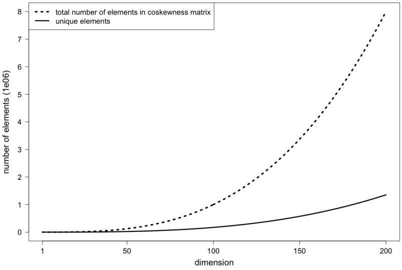
**Issues:**

- computational
- statistical

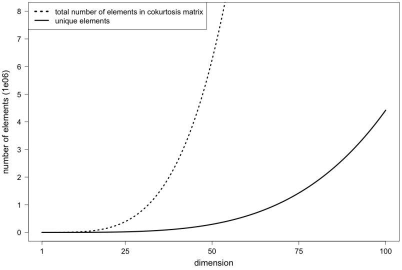
# SAMPLE ESTIMATORS

## NUMBER OF UNIQUE ELEMENTS

(a) Coskewness




(b) Cokurtosis



# SAMPLE ESTIMATORS

## NUMBER OF UNIQUE ELEMENTS

**Speed gain:**

current implementation in `PerformanceAnalytics` (Fortran) vs. estimation of only unique elements (`RcppArmadillo`)

- Coskewness matrix:

| $p$ | 10 | 20 | 30 | 50 | 100 |
|---|---|---|---|---|---|
| Absolute time (ms.) | 0.40 | 1.20 | 4.10 | 18.90 | 183.10 |
| Relative improvement (%) | 275.00 | 608.30 | 614.60 | 406.90 | 880.20 |

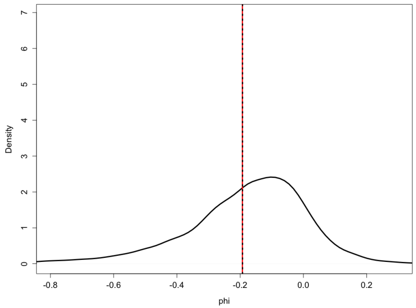
- Cokurtosis matrix:

| $p$ | 10 | 20 | 30 | 50 | 100 |
|---|---|---|---|---|---|
| Absolute time (ms.) | 0.90 | 11.10 | 65.30 | 346.50 | 4284.40 |
| Relative improvement (%) | 1200.00 | 993.70 | 1831.20 | 2908.20 | 3088.50 |

Comparison made with the package `rbenchmark`

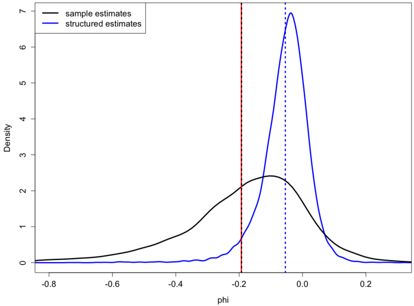
# SAMPLE ESTIMATORS

## STATISTICAL ISSUES




- consistent
- large mean squared error (MSE)
- many unique elements (error maximization)

# STRUCTURED ESTIMATION




Assume a certain data-generating process, e.g. independence, factor model, . . .

- imposes a certain structure on the covariance, coskewness and cokurtosis matrices
- reduces number of free parameters
- potentially reduces the MSE (when the model is not too misspecified)

# LINEAR SHRINKAGE

- Introduced by Ledoit and Wolf (2003) for covariance estimation.
- Extended to coskewness and cokurtosis estimation in Martellini and Ziemann (2010).
- Improved shrinkage estimation of the coskewness matrix in Boudt, Cornilly and Verdonck (2017a). Contributions:
  - Unbiased and consistent estimators leading to lower MSE,
  - optimized target matrices,
  - possibility for multi-target shrinkage (using multiple targets at once).

## LINEAR SHRINKAGE

$$\widehat{\boldsymbol{\Phi}}^{\mathsf{ST}}(\lambda) = (1-\lambda)\widehat{\boldsymbol{\Phi}} + \lambda\widehat{\boldsymbol{T}}, \qquad \lambda \in [0,1],$$

where

- $\widehat{\boldsymbol{\Phi}}$ is the unbiased sample estimator,
- $\widehat{\boldsymbol{T}}$ is a structured coskewness matrix

Aim: choose $\lambda$ as optimal trade-off between

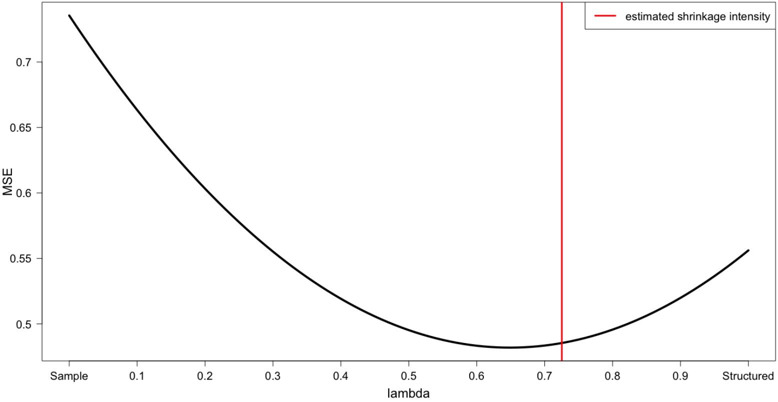
- unbiasedness and large estimation variance of $\widehat{\boldsymbol{\Phi}}$,
- biased but smaller estimation variance of $\widehat{\boldsymbol{T}}$.

Approach: minimize the MSE,

$$\lambda^{\star} = \arg\min_{\lambda\in[0,1]} \mathbb{E}\left[\|(1-\lambda)\widehat{\boldsymbol{\Phi}} + \lambda\widehat{\boldsymbol{T}} - \boldsymbol{\Phi}\|^2\right]$$

# LINEAR SHRINKAGE

Figure: Mean Squared Error



# NEAREST COMOMENT ESTIMATION

Suppose that the data follows a multi-factor model,

$$\boldsymbol{X} = \boldsymbol{B}\boldsymbol{F} + \varepsilon,$$

then (see Boudt, Lu and Peeters (2015)),

$$\begin{aligned}
\boldsymbol{\Sigma}_{\boldsymbol{X}} &= \boldsymbol{B}\boldsymbol{\Sigma}_{\boldsymbol{F}}\boldsymbol{B}' + \boldsymbol{\Delta} \\
\boldsymbol{\Phi}_{\boldsymbol{X}} &= \boldsymbol{B}\boldsymbol{\Phi}_{\boldsymbol{F}}(\boldsymbol{B}' \otimes \boldsymbol{B}') + \boldsymbol{\Omega} \\
\boldsymbol{\Psi}_{\boldsymbol{X}} &= \boldsymbol{B}\boldsymbol{\Psi}_{\boldsymbol{F}}(\boldsymbol{B}' \otimes \boldsymbol{B}' \otimes \boldsymbol{B}') + \boldsymbol{\Gamma}
\end{aligned}$$

- Factors are **observed**: structured estimation as in Boudt, Lu and Peeters (2015)
- Factors are **unobserved**: nearest comoment estimation as in Boudt, Cornilly and Verdonck (2017b) (working paper)

# Conclusion

# CONCLUSION

- Higher order moments matter
- Many practical applications
- Estimation matters more!
- Coming soon: update to `PerformanceAnalytics`
  - improve computational efficiency of the higher order moments
  - implementation of several state-of-the-art estimators

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