Rearrangement Algorithm and Maximum Entropy

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Introduction

- An important problem in Finance:

  Given marginal distributions of random variables $X_1, \ldots, X_d$ *and* of their weighted sum $I = \omega_1 X_1 + ... + \omega_d X_d$, can we infer dependence among the variables?

- For example, if X_j's are stock prices and I is the stock index, we can estimate their risk-neutral densities from traded options using Breeden and Litzenberger (1978) result:

$$f(K) = \frac{\partial^2 C(K)}{\partial K^2}.$$

- Using forward-looking risk-neutral densities implied by option prices, can we infer the **dependence**?

Introduction

- In this paper, we study properties of the Block Rearrangement Algorithm (BRA) in the context of inferring dependence among variables given their marginal distributions and the distribution of the sum.
- Although there are typically infinitely many theoretical solutions, we show that BRA yields solutions that are close to each other and exhibit maximum entropy.
- Thus, BRA is a stable algorithm for inferring dependence and its solution is economically meaningful.

Setup

- Inputs: d random variables $X_1 \sim F_1$, ..., $X_d \sim F_d$.
- Goal: look for a dependence such that the variance of sum $S = X_1 + ... + X_d$ is minimized.
- Assume that each X_j is sampled into n equiprobable values, i.e., we consider the realizations $x_{ij} := F_j^{-1}(\frac{i-0.5}{n})$ and arrange them in an $n \times d$ matrix:

$$\mathbf{X} = [X_1, \ldots, X_d] = \begin{bmatrix} x_{11} & x_{12} & \ldots & x_{1d} \\ x_{21} & x_{22} & \ldots & x_{2d} \\ \vdots & \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & \ldots & x_{nd} \end{bmatrix}$$

- Want to rearrange elements x_{ij} (by columns) so that the variance of row sums is minimized.
- This is an NP complete problem. Brute force search requires checking $(n!)^{(d-1)}$ rearrangements!

Rearrangement Algorithm (RA)

- Greedy algorithm developed in Puccetti and Rüschendorf (2012) and Embrechts, Puccetti, and Rüschendorf (2013):

  1. For $j = 1, \ldots, d$, make the j^{th} column anti-monotonic with the sum of the other columns.
  2. If there is no improvement in var $\left(\sum_{k=1}^{d} X_k\right)$, output the current matrix $\mathbf{X}$, otherwise return to step 1.

- Step 1 ensures that the columns before rearranging (X_k) and after rearranging $(\widetilde{X}_k)$ satisfy

$$\text{var}\left(\sum_{k=1}^{d} X_k\right) \geqslant \text{var}\left(\sum_{k=1}^{d} \widetilde{X}_k\right).$$

Block Rearrangement Algorithm (BRA)

- When $d > 3$, the standard RA can be improved by considering blocks, Bernard and McLeish (2014), Bernard, Rüschendorf, and Vanduffel (2014):

  1. Select a random sample of n_{sim} possible partitions of the columns $\{1, 2, \ldots, d\}$ into two non-empty subsets $\{I_1, I_2\}$.
  2. For each of the n_{sim} partitions, create block matrices $\mathbf{X}_1$ and $\mathbf{X}_2$ with corresponding row sums S_1 and S_2 and rearrange rows of $\mathbf{X}_2$ so that S_2 is anti-monotonic to S_1.
  3. If there is no improvement in var $\left(\sum_{k=1}^{d} X_k\right)$, output the current matrix $\mathbf{X}$, otherwise, return to step 1.

- When d is reasonably small, we can take $n_{sim} = 2^{d-1} - 1$ (all non-trivial partitions are considered). Otherwise, randomize.

Inferring Dependence

- Inputs: d random variables $X_1 \sim F_1$, ..., $X_d \sim F_d$ and their sum $S \sim F_S$.
- Assume that each X_j and S are sampled into n equiprobable values, arranged in an $n \times (d+1)$ matrix:

$$\mathbf{M} = [X_1, \dots, X_d, -S] = \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1d} & -s_1 \\ x_{21} & x_{22} & \dots & x_{2d} & -s_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{nd} & -s_n \end{bmatrix}.$$

- Apply BRA on the augmented matrix $\mathbf{M}$.
- When row sums of the rearranged matrix are close to zero, a **compatible dependence** has been found.

Simulation Exercise: Gaussian Case

- Gaussian margins $X_j \sim N[0, \sigma_j^2]$ and Gaussian sum $S \sim N[0, \sigma_S^2]$.
- Number of components d ranges from 3 to 10.
- Standard deviations σ_i are linearly decreasing from 1 to $1/d$.
- Set σ_S such that $\rho_{imp} = 0.8$.
- Discretization level n from 1,000 to 10,000.
- Run BRA $K = 500$ times.
- Find that for each k the inferred dependence is close to the one with the maximum entropy, which
  - has a Gaussian copula,
  - maximizes the determinant of the correlation matrix for X_j.

Maximum Determinant and Maximum Entropy

- Entropy refers to disorder of a system, Shannon (1948).
- Let f be the density of a multivariate distribution of $(X_1, ..., X_d)$, then the entropy is

$$H(X_1, ..., X_d) = -\mathsf{E}(\log(f(X_1, ..., X_d))).$$

Proposition: Maximum entropy

The entropy of the multivariate distribution of $(X_1, ..., X_d)$ with Gaussian margins and invertible correlation matrix R satisfies

$$H(X_1, .., X_d) \leq \frac{d}{2}(1 + \ln(2\pi)) + \frac{1}{2}\sum_{i=1}^{d} \ln\left(\sigma_i^2\right) + \frac{1}{2}\ln(\det(R))$$

where the equality holds *iff* $(X_1, ..., X_d)$ is multivariate Gaussian.

Stability of BRA

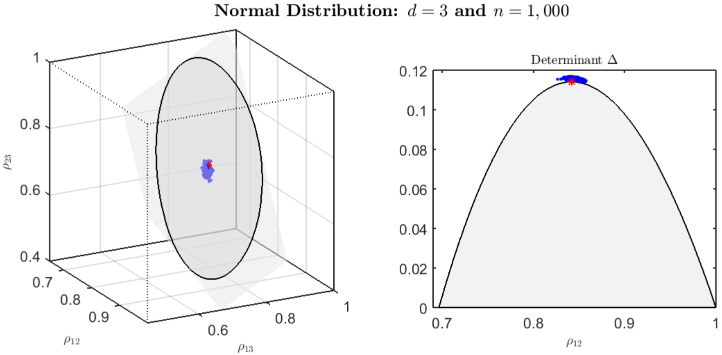
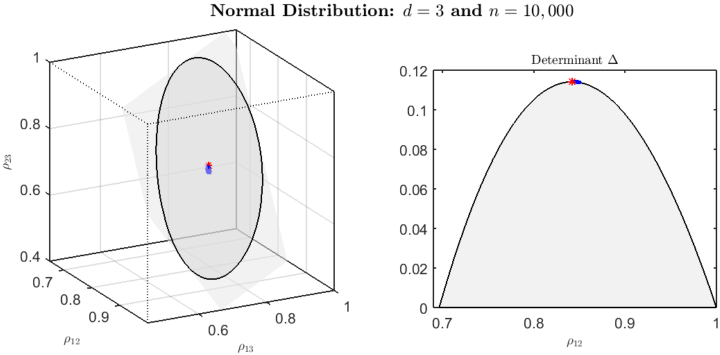


Figure: $K = 500$ blue dots correspond to different runs of BRA. Shaded gray area is set of feasible solutions; red star is maximal correlation matrix R_M (=maximum entropy). Left panel: realized correlations ρ_{12}, ρ_{13}, and ρ_{23}. Right panel: relation of determinant Δ versus ρ_{12}.

Stability of BRA



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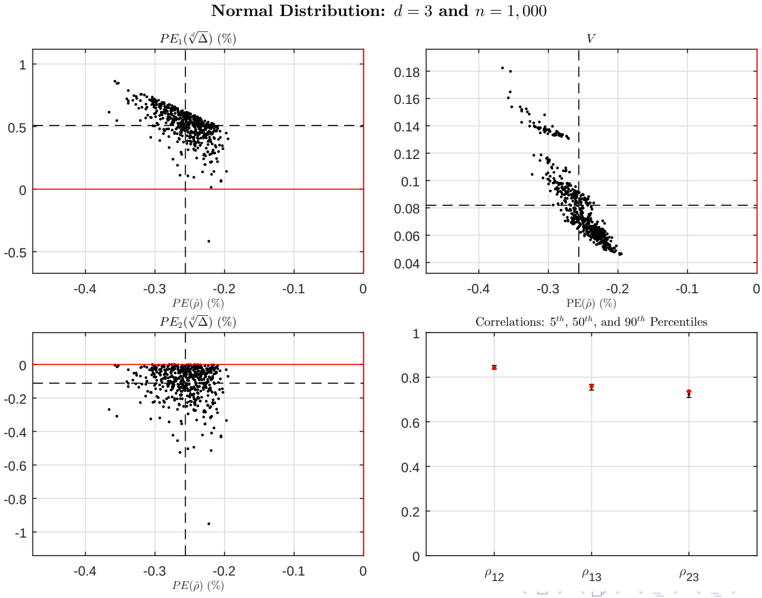
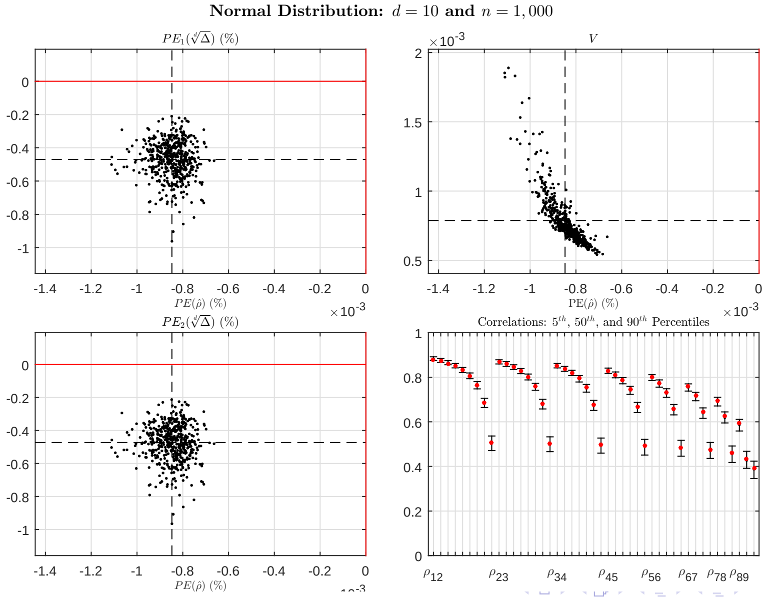
Recovering Pairwise Correlations



Recovering Pairwise Correlations



Robustness to Initial Conditions

- Start from a particular candidate solution.
- Introduce small noise, by randomly swapping 0.2% of rows:
  - 2 rows out of 1,000,
  - 6 rows out of 3,000,
  - 20 rows out of 10,000.
- Check where $K = 500$ runs of BRA converge.

Robustness to Initial Conditions

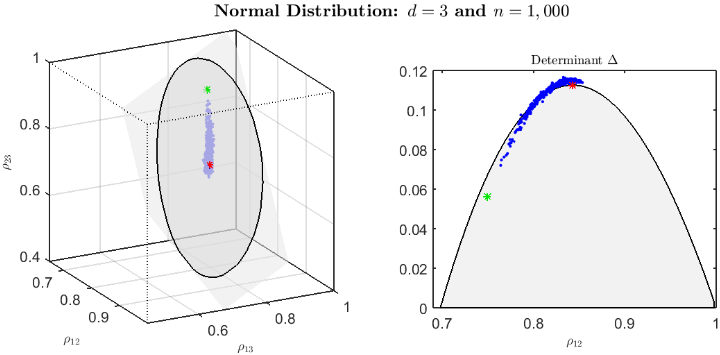


Figure: $K = 500$ blue dots correspond to different runs of BRA. Each run starts at a particular solution (green star), but with 2 random rows swapped. Shaded gray area is set of feasible solutions; red star is maximal correlation matrix R_M (=maximum entropy).

Robustness to Initial Conditions

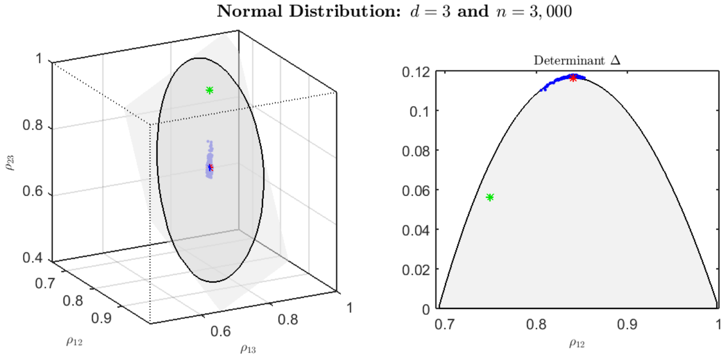


Figure: $K = 500$ blue dots correspond to different runs of BRA. Each run starts at a particular solution (green star), but with 6 random rows swapped. Shaded gray area is set of feasible solutions; red star is maximal correlation matrix R_M (=maximum entropy).

Robustness to Initial Conditions

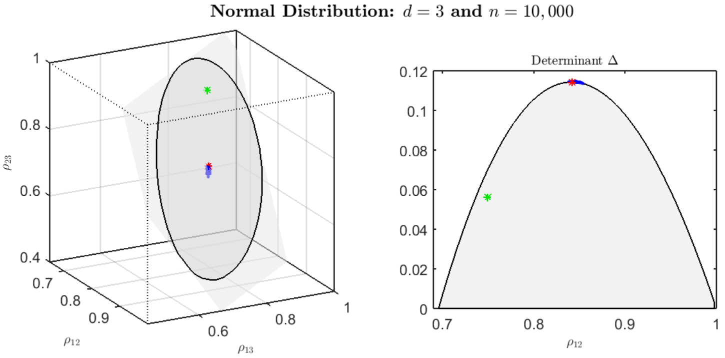


Figure: $K = 500$ blue dots correspond to different runs of BRA. Each run starts at a particular solution (green star), but with 20 random rows swapped. Shaded gray area is set of feasible solutions; red star is maximal correlation matrix R_M (=maximum entropy).

Conclusions

- Robust to non-Gaussian distributions (e.g., Multivariate Skewed-t).
- Does not hold for the standard RA.
- Applications:
  - Pricing multivariate options (basket, exchange, spread, etc.),
  - **Forward looking** indicators of implied dependence, measures of **tail risk**,
  - **Down** and **Up** implied correlation,
  - Optimal portfolios.
- Paper is available on SSRN.