A Nonparametric Estimate of the Risk-Neutral Density and Its Applications

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May 2017

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Piece-wise constant nonparametric approach - Setup 1

- Constraints:
  1. Non-negativity: $a_1, a_2, \ldots, a_{q+1} \geq 0$
  2. Unity-integration:

$$\sum_{l=1}^{q+1} a_l \ log \frac{K_l}{K_{l-1}} = 1$$

- Optimized objective function in terms of $q+1$ parameters:

$$\underset{a_1,\ldots,a_{q+1}}{\text{argmin}} \{ \sum_{j' \in \mathcal{C}} (\frac{C_{j'} - \tilde{C}_{j'}}{\tilde{C}_{j'}})^2 + \sum_{i \in \mathcal{P}} (\frac{P_i - \tilde{P}_i}{\tilde{P}_i})^2 \}$$

subject to the above two constraints.

where $\tilde{P}_i$ and $\tilde{C}_{j'}$ are market option prices;

P_i and $C_{j'}$ are fair option prices.

WLS using all-option fit

WLS	$7 \sim 14$	$17 \sim 31$	$81 \sim 94$	$171 \sim 199$	$337 \sim 393$	$502 \sim 592$	$670 \sim 790$
$L_r(OTM)^c$	0.0751	0.0701	0.0241	0.0377	0.0367	0.0342	0.0440
$L_a(OTM)^c$	0.1063	0.1212	0.2460	2.7290	3.4660	3.5504	6.1585
$L_r(ITM)^c$	0.0212	0.0196	0.0155	0.0896	0.0445	0.0403	0.0537
$L_a(ITM)^c$	2.8701	3.0760	4.5941	10.4246	10.2022	15.3275	10.7337
	$7 \sim 14$	$17 \sim 31$	$81 \sim 94$	$171 \sim 199$	$337 \sim 393$	$502 \sim 592$	$670 \sim 790$
$L_r(OTM)^p$	0.0767	0.0644	0.0344	0.0468	0.0327	0.0444	0.0518
$L_a(OTM)^p$	0.1125	0.0878	0.1903	1.8165	2.0243	2.7545	5.1333
$L_r(ITM)^p$	0.0226	0.0202	0.0333	0.0623	0.0636	0.0701	0.0704
$L_a(ITM)^p$	2.2309	2.3946	4.6722	5.3686	8.7092	17.4991	11.9397

Table 1: Prediction errors for all options under WLS using all options across different numbers of days to expiration

Comparison with cubic spline methods - Random 200 pairs

	$7 \sim 14$	$17 \sim 31$	$81 \sim 94$	$171 \sim 199$	$337 \sim 393$	$502 \sim 592$	$670 \sim 790$
$L_r(OTM)^p$	4.3636	3.0586	2.9058	16392.0113	1002.5010	90.2846	266.3731
$L_a(OTM)^p$	3.3993	1.7206	3.4074	190668.5547	21012.3756	3996.6323	24201.2311
$L_r(ITM)^p$	0.2677	0.1566	0.2006	952.0472	359.5709	298.3378	0.1482
$L_a(ITM)^p$	5.3896	4.8317	13.6291	89828.9872	111633.9924	57267.4141	21.8593
	$7 \sim 14$	$17 \sim 31$	$81 \sim 94$	$171 \sim 199$	$337 \sim 393$	$502 \sim 592$	$670 \sim 790$
$L_r(OTM)^c$	8.8340	2.5977	14.0896	35153.2759	2302.5577	61680.2524	3823.3592
$L_a(OTM)^c$	5.1684	2.7804	64.7542	231577.1938	176428.5863	882193.0322	21758.2561
$L_r(ITM)^c$	0.3388	0.1643	0.8858	1129.9532	1330.0822	2032.7718	147.4041
$L_a(ITM)^c$	7.7784	5.5999	35.7376	160585.4643	355420.4223	637677.2702	71984.7514

Table 2: Error results -**cubic spline**

	$7 \sim 14$	$17 \sim 31$	$81 \sim 94$	$171 \sim 199$	$337 \sim 393$	$502 \sim 592$	$670 \sim 790$
$L_r(OTM)^p$	0.0821	0.0576	0.0195	0.0142	0.0041	0.0133	0.0337
$L_a(OTM)^p$	0.1058	0.0928	0.2323	0.4836	0.5745	0.9600	0.9866
$L_r(ITM)^p$	0.0306	0.0260	0.0202	0.0136	0.0145	0.0188	0.0423
$L_a(ITM)^p$	1.4264	1.3410	1.6377	1.3798	2.4415	2.5551	3.8917
	$7 \sim 14$	$17 \sim 31$	$81 \sim 94$	$171 \sim 199$	$337 \sim 393$	$502 \sim 592$	$670 \sim 790$
$L_r(OTM)^c$	0.0730	0.0502	0.0158	0.0164	0.0174	0.0201	0.0147
$L_a(OTM)^c$	0.1021	0.1154	0.3746	0.6520	1.4593	2.3057	1.2442
$L_r(ITM)^c$	0.0315	0.0271	0.0213	0.0168	0.0097	0.0204	0.0234
$L_a(ITM)^c$	4.0856	2.8867	5.0610	4.7761	1.1971	3.3539	4.1070

Table 3: Error results -**Our approach**

Applications of risk-neutral density estimate

- Recover risk-neutral density
  - Very consistent with the common guess of the risk-neutral density shape

> R.Input: allop.RND(oplist, Klist, optypelist, rate, tau)
> R.Output: RND and knots

Applications of risk-neutral density estimate - Continued

- Unveil investment opportunities

  - Able to recognize some options on the markets that are under or above estimated and explore profitable investment opportunities for investors

> R.Input: bootCI.function(oplist, Klist, optypelist, rate, tau, n, alpha)
> R.Output: confidence interval for each option price

Applications of risk-neutral density estimate - Continued

- Provide a fair price for any derivative with the same time to expiration, say options

> R.Input: allfit.pred.oneK(oplist, Klist, optypelist, rate, tau, optype, K)
> R.Output: fair price for options using leave one out cross validation

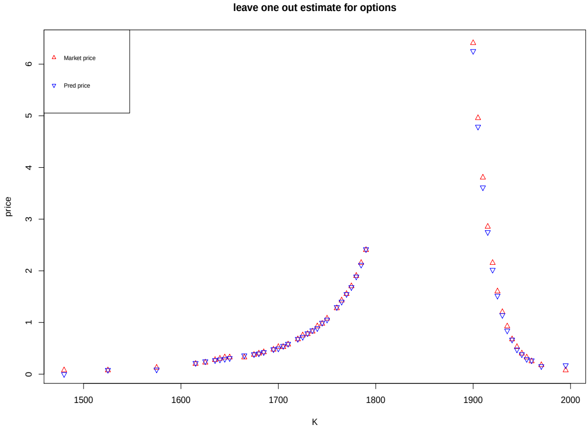

Figure 1: An example of leave-one-out cross validation performance of our approach under LS setup for unveil investment opportunities

Variance swap

- **What is variance swap?**
  A financial product which allows investors to trade future realized (or historical) variance against current implied variance.

- **What is variance?**
  It measures the degree of variation of a trading price series over time as measured by the variance of returns

- **Why variance swap important?**
  - Easy and straightforward to hedge risks
  - Pure exposure to the variance of an underlying asset, no directional risk
  - Low transaction costs
  - Liquid across major equity indices and large cap stocks
  - Systematically selling variance has historically been profitable

Pricing performance

1. Method 1 (OP) - Moment-based method: use option data to generate as our theoretical prices
2. Method 2 (VF) - Calibration method: replicate CBOE traded variance future to create historical prices
3. Method 3 (True): Use the true historical market prices of the underlying asset of S&P 500, and calculate the "True" realized variance of corresponding period by taking summation of squares of log returns to further provide reference

Use ratios to present the results:
- OP/True, VF/True, and OP/VF.

Results


OPTrue and VFTrue vs Remaining Cdays
Ratio
1.5
2.0
2.5
3.0
3.5
365
400
500
600
700
800
900
Remaining calender days of contract
VFTrue
OPTrue

Conclusion

- Our method can be very consistent in predicting price of variance swap that has expiration more than one year.

Benefit: Better predict the variation of fair price of contract and be profitable

Selected references

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Thank you!